not have all four factors.* The former are called normal forms. It is shown that the latter are capable of being reduced by means of one or more of four kinds of reduction to normal forms of resultants $R_{\mu,\nu}$ of lower order and are called reducible forms, or else are *completely* reducible to a_0 or b_0 (in reality to unity), and are then called completely reducible forms.

(2). *Formulas.*

The normal forms have according to (1) the formula,

$$N_{m,n}=a_0 a_{\kappa_2}\ldots.a_{\kappa\ n-1}a_m b_0 b_{\lambda_2}\ldots.b_{\lambda\ m-1}\ b_n=(a_0)^{p_1}(a_{r_2})^{p_2}$$

$$\ldots(a_{r\ \mu-1})^{p_\mu-1}(a_m)^{p_m}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_{\nu-1}})^{q_\nu-1}(b_n)^{q_\nu}.$$

The normal form $N_{\mu,\nu}$ of the resultant $R_{\mu,\nu}$, to which a reducible term

$$P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$$

of $R_{m,n}$ is reduced, (by means of reductions to be stated in 2) has the formula

$$N_{\mu,\nu}=(a_{r_\lambda-\rho})^{p_\lambda-\alpha}(a_{r_{\lambda+1}-\rho})^{p_{\lambda+1}}\ldots(a_{r_\tau-\rho})^{p_\tau-r}(b_{s_\kappa-\sigma})^{q_\kappa-\beta}(b_{s_{\kappa+1}-\sigma})^{q_{\kappa+1}}\ldots(b_{s_\pi-\sigma})^{q_\pi-\delta},$$

where $r_\lambda=\rho$, $r_\tau=\mu+\rho$, $s_\kappa=\sigma$, $s_\pi=\nu+\sigma$, and where at least one of the numbers α, β, and at least one of the numbers r, δ is zero. Similarly, but without altering its value, the coefficient

$$C_{m,n}=(r_1)^{p_1}(r_2)^{p_2}\ldots.(r_\mu)^{p_\mu}\ |\ (s_1)^{q_1}(s_2)^{q_2}\ldots.(s_\nu)^{q_\nu}$$

of $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.b_{s_\nu})^{q_\nu}$

is reduced in form to $C_{\mu,\nu}$, so that $C_{m,n}=C_{\mu,\nu}$, or

$$(r_1)^{p_1}(r_2)^{p_2}\ldots.(r_\mu)^{p_\mu}\ |\ (s_1)^{q_1}(s_2)^{q_2}\ldots.s_\nu)^{q_\nu}$$

$$=(r_\lambda-\rho)^{p_\lambda-\alpha}(r_{\lambda+1}-\rho)^{p_{\lambda+1}}\ldots(r_\tau-\rho)^{p_\tau-r}\ |\ (s_\kappa-\sigma)^{q_\kappa-\beta}(s_{\kappa+1}-\sigma)^{q_{\kappa+1}}\ldots(s_\pi-\sigma)^{q_\pi-\delta}$$

The completely reducible forms of $R_{m,n}$ have the general formula :

$$v_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_0)^{r_1}(b_{p_1})^{r_2-r_1}(b_{p_1+p_2})^{r_3-r_2}\ldots.(b_{p_1+p_2+\ldots.p_\mu})^{m-r_\mu}.$$

2. The Four Kinds of Reduction.

The before-mentioned reductions are attained in the following ways :

(1). *The first reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_n)^{q_\nu}$, a form of $R_{m,n}$, which contains the factor b_n, but not a_m, by $(b_n)^{m-r_\mu}$, $P_{m,n}$ is reduced to $P_{r_\mu,\ n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.b_n{}^{q_\nu-m+r_\nu}$, a form of a lower resultant $R_{r_\mu,n}$.

*It is a fundamental condition that every form $P_{m,n}$ must contain at least one of the factors a_0, b_0, and at least one of the factors a_m, b_n.

(2). *The second reduction.*

By dividing $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_m)^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$ of $R_{m,n}$, which has the factor a_m but not b_n, by $(a_m)^{n-s_\nu}$, we obtain $P_{m,s_\nu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_m)^{p_\mu-n+s_\nu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$, a form of the resultant R_{m,s_ν} of lower order than $R_{m,n}$.

(3). *The third reduction.*

Here $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$ (with $r_1\neq 0$) is reduced to $P_{m-r_1,n}=(a_0)^{p_1}(a_{r_2-r_1})^{p_2}\ldots.(a_{r_\mu-r_1})^{p_\mu}(b_0)^{q_1-r_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$, a form of the resultant $R_{m-r_1,n}$, by dividing by $(b_0)^{r_1}$ and diminishing the subscripts of the a's by r_1.

(4). *The fourth reduction.*

In this case $P_{m,n}=(a_0)^{p_1}(a_{r_2})^{p_2}\ldots\ldots(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots\ldots(b_{s_\nu})^{q_\nu}$, $(s_1\neq 0)$, is reduced to $P_{m,n-s_1}=(a_0)^{p_1-s_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_0)^{q_1}(b_{s_2-s_1})^{q_2}\ldots.(b_{s_\nu-s_1})^{q_\nu}$, of the resultant $R_{m,n-s_1}$, by division by $(a_0)^{s_1}$, and diminution of the subscripts of the b's by s_1.

3. The Four Kinds of Derivation.

Conversely we may start with forms of lower resultants and by four kinds of derivation attain to forms of higher resultants.

(1). *The first kind of derivation.*

By multiplying $P_{m,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$ by $b_n{}^q$, we obtain $P_{m+q,n}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}b_n{}^q$, a form of the resultant $R_{m+q,n}$.

(2). *The second kind of derivation.*

Similarly from $P_{m,n}$ we obtain $P_{m,n+p}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots.(a_{r_\mu})^{p_\mu}(a_m)^p(b_{s_1})^{q_1}(b_{s_2})^{q_2}\ldots.(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m,n+p}$, by multiplication by $a_m{}^p$.

(3). *The third kind of derivation.*

In this case we obtain $P_{m+q,n}=(a_{r_1+q})^{p_1}(a_{r_2+q})^{p_2}\ldots.(a_{r_\mu+q})^{p_\mu}(b_0)^q(b_{s_1})^{q_1}\ldots.(b_{s_\nu})^{q_\nu}$, of the resultant $R_{m+q,n}$, from $P_{m,n}$ by multiplication by $b_0{}^q$ and increase of the subscripts of the a's by q.

(4). *The fourth kind of derivation.*

Here we multiply $P_{m,n}$ by $a_0{}^p$ and increase the subscripts of the b's by p, and obtain $P_{m,n+p}=(a_0)^p(a_{r_1})^{p_1}\ldots.(a_{r_\mu})^{p_\mu}(b_{s_1+p})^{q_1}(b_{s_2+p})^{q_2}\ldots.(b_{s_\nu+p})^{q_\nu}$, of the resultant $R_{m,n+p}$.

4. Recurrence Formula for the Normal Coefficients.

For the calculation of the coefficients of the normal forms we use the formula

$$p_1\times 0^{p_1}(r_2)^{p_2}\ldots.m^{p_\mu}\mid 0^{q_1}(s_2)^{q_2}\ldots.n^{q_\nu}=\sum_{\lambda=2}^{\lambda=\nu}(-1)^{s_\lambda+1}(c_\lambda+1)\times s_\lambda\,0^{p_1-1}(r_2)^{p_2}\ldots.m^{p_\mu}\mid(s_\lambda)^{-1}0^{q_1+1}(s_2)^{q_2}\ldots.n^{q_\nu},$$

where c_λ is the exponent of s_λ in the expression $0^{p_1-1}(r_2)^{p_2}\ldots.m^{p_\mu}$. It is a recurrence formula and serves for the calculation of the coefficient of the normal form $(a_0)^{p_1}(a_{r_2})^{p_2}\ldots.(a_m)^{p_\mu}(b_0)^{q_1}(b_{s_2})^{q_2}\ldots.(b_n)^{q_\nu}$ from earlier calculated coeffi-

cients of simpler forms. In it every coefficient on the right hand is to be reduced to its simplest form according to 1, (2).

5. THE RESULTANT $R_{m,n}$ IN TERMS OF SYMMETRIC FUNCTIONS.

(1). *The expressions.*

The resultant $R_{m,n}$ of f and ϕ may be written in the forms,

$$(-1)^{mn} b_0{}^m f(\beta_1) f(\beta_2) \dots f(\beta_n) =$$

$$(-1)^{mn} b_0{}^n [a_1 \beta_1{}^m - a_1 \beta_1{}^{m-1} + \dots + (-1)^m a_m][a_0 \beta_2{}^m - a_1 \beta_1{}^{m-1} + \dots + (-1)^m a_m]$$

$$\dots [a_0 \beta_n{}^m - a_1 \beta_n{}^{m-1} + \dots + (-1)^m a_m] = a_0{}^n \phi(\alpha_1) \phi(\alpha_2) \dots \phi(\alpha_n)$$

$$= a_0{}^n (b_0 \alpha_1{}^n + \dots + b_n)(b_0 \alpha_2{}^n + \dots + b_n) \dots (b_0 \alpha_m{}^n + \dots + b_n).$$

(2). *The coefficient* $(m-\varkappa_1)(m-\varkappa_2) \dots (m-\varkappa_n) \mid 0^{\lambda_0} 1^{\lambda_1} \dots n^{\lambda_n}$.

The coefficient of $a_{m-\varkappa_1} a_{m-\varkappa_2} \dots a_{m-\varkappa_n}$, using the first form of $R_{m,n}$, is $(-1)^{\varkappa_1+\varkappa_2+\dots+\varkappa_n} b_0{}^m \Sigma \beta_1{}^{\varkappa_1} \beta_2{}^{\varkappa_2} \dots \beta_n{}^{\varkappa_n}$. In this again the coefficient $(m-\varkappa_1)(m-\varkappa_2) \dots (m-\varkappa_n) \mid 0^{\lambda_0} 1^{\lambda_1} \dots n^{\lambda_n}$ of the product $a_{m-\varkappa_1} a_{m-\varkappa_2} \dots a_{m-\varkappa_n} b_0{}^{\lambda_0} b_1{}^{\lambda_1} \dots b_n{}^{\lambda_n}$ in $R_{m,n}$, is equal numerically to the coefficient of $b_0{}^{\lambda_0} b_1{}^{\lambda_1} \dots b_n{}^{\lambda_n}$ in $b_0{}^n \Sigma \beta_1{}^{\varkappa_1} \beta_2{}^{\varkappa_2} \dots \beta_n{}^{\varkappa_n}$, that is,

$$(m-\varkappa_1)(m-\varkappa_2) \dots (m-\varkappa_n) \mid 0^{\lambda_0} 1^{\lambda_1} \dots n^{\lambda_n} = (-1)^{\varkappa_1+\varkappa_2+\dots+\varkappa_n}$$

$$\binom{0^{\lambda_0} 1^{\lambda_1} \dots \dots n^{\lambda_n}}{b_0{}^m \beta_1{}^{\varkappa_1} \beta_2{}^{\varkappa_2} \dots \beta_n{}^{\varkappa_n}}, \text{ where } \binom{0^{\lambda_0} 1^{\lambda_1} \dots \dots n^{\lambda_n}}{b_0{}^m \beta_1{}^{\varkappa_1} \beta_2{}^{\varkappa_2} \dots \beta_n{}^{\varkappa_n}}$$

denotes the coefficient of $b_0{}^{\lambda_0} b_1{}^{\lambda_1} \dots b_n{}^{\lambda_n}$ in $b_0{}^m \Sigma \beta_1{}^{\varkappa_1} \beta_2{}^{\varkappa_2} \dots \beta_n{}^{\varkappa_n}$.

Using the second form of the resultant, the coefficient of $b_0{}^{\lambda_0} b_1{}^{\lambda_1} \dots b_n{}^{\lambda_n}$ is

$$a_0{}^n \Sigma (\alpha_1 \alpha_2 \dots \alpha_{\lambda_0})^n (\alpha_{\lambda_0+1} \dots \alpha_{\lambda_0+\lambda_1})^{n-1} \dots (\alpha_{\lambda_0+\lambda_1+\dots+\lambda_{n-1}} \dots \alpha_{\lambda_0+\dots+\lambda_n})^0.$$

This last symmetric function we will express symbolically by

$$a_0{}^n \Sigma (\alpha \lambda_0)^n (\alpha \lambda_1)^{n-1} (\alpha \lambda_2)^{n-2} \dots (\alpha \lambda_n)^0,$$

where $(\alpha \lambda_r)^{n-r}$ means that λ_r roots (α) have the exponent $n-r$.

In this symmetric function the coefficient

$$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\cdots\cdot n^{\lambda_n}$$

of the product

$$a_{m-\varkappa_1}a_{m-\varkappa_2}\ldots.a_{m-\varkappa_n}b_0{}^{\lambda_0}b_1{}^{\lambda_1}\ldots.b_n{}^{\lambda_n}$$

in $R_{m,n}$ is equal numerically to the coefficient of $a_{m-\varkappa_1}a_{m-\varkappa_2}\ldots.a_{m-\varkappa_n}$ in $a_0{}^n\Sigma(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots.(\alpha\lambda_n)^0$, that is

$$(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n) \mid 0^{\lambda_0}1^{\lambda_1}\ldots.n^{\lambda_n}=(-1)^{mn+\varkappa_1+\varkappa_2+\ldots.\varkappa_n}$$

$$\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n)\\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots.(\alpha\lambda_n)^0\end{pmatrix}.$$

(3). *Relation between coefficients of terms in symmetric functions.*

By (2) we have two expressions for the coefficient of a term of $R_{m,n}$. By equating these expressions, we obtain a relation between the coefficients of terms of two symmetric functions, namely :

$$\begin{pmatrix}0^{\lambda_0}1^{\lambda_1}\ldots\ldots\ldots n^{\lambda_n}\\ b_0{}^m\beta_1{}^{\varkappa_1}\beta_2{}^{\varkappa_2}\ldots\ldots\beta_n{}^{\varkappa_n}\end{pmatrix}=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots.(m-\varkappa_n)\\ a_0{}^n(\alpha\lambda_0)^n(\alpha\lambda_1)^{n-1}\ldots.(\alpha\lambda_n)^0\end{pmatrix},$$

which is seen to be true whether f is written with alternating sign or not.

With this paragraph the statements concerning the resultant theory, so far as they relate to this paper are finished. It is now proposed to develop a theory for symmetric functions similar to that sketched for the resultant, and to point out the correspondences between them. Moreover, the theory is developed independently of the results for the resultant, with the exception of the last formula, which will be used.

[To be continued.]

THE COLLECTED MATHEMATICAL PAPERS OF ARTHUR CAYLEY.

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[4to. 13 Vols., each $6.25. Supplementary Vol. containing Titles of Papers and Index, $2.50. Macmillan.]

This republication by the Cambridge University Press of Cayley's papers in collected form is the most fitting monument of his splendid fame. He must ever rank as one of the greatest mathematicians of all time. Cayley exceedingly appreciated this action of the Syndics of the Press, and seven of the large quarto volumes appeared under his own editorship. As to what these 13 volumes contain, it seems vain to attempt even a summary. They cover the whole range of pure mathematics, algebra, analysis, mathematical astronomy, dynamics, and in particular groups, quadratic forms, quantics, etc., etc.

Though abreast of Sylvester as an analyst, he was, what Sylvester was not, also a geometer. Again and again we find the pure geometric methods of Poncelet and Chasles, though perhaps not full assimilation of that greater one than they, who has now absorbed them—Von Staudt. Cayley not only made additions to every important subject of pure mathematics, but whole new subjects, now of the most importance, owe their existence to him. It is said that he is actually now the author most frequently quoted in the living world of mathematicians.

His name is perhaps most closely linked with the word *invariant*, due to his great brother-in-arms, Sylvester. Boole in 1841 had shown the invariance of all discriminants and given a method of deducing some other such functions. This paper of Boole's suggested to Cayley the more general question, to find "all the derivations of any number of functions which have the property of preserving their form unaltered after any linear transformation of the variables." His first results, relating to what we now call invariants, he published in 1845. A second set of results, relating to what Sylvester called covariants, he published in 1846. Not until four or five years later did Sylvester take up this matter, but then came such a burst of genius that after his series of publications in 1851-54 the giant theory of Invariants and Covariants was in the world completely equipped.

The check came when Cayley, in his second Memoir on Quantics, came to the erroneous conclusion that the number of the asyzygetic invariants of binary quantics beyond the sixth order was infinite, "thereby," as Sylvester says, "arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion." The passages supposed to prove this are marked "*incorrect*" in the Collected Mathematical Papers. But this error was not corrected until 1869, [Crelle, vol. 69., pp. 323-354] by Gordan in his Memoir [dated 8th June, 1868], 'Beweis dass jede Covariante und Invariante einer

binaeren Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.'

Cayley at once returned to the question, found the source of his mistake, the unspected and so neglected interdependence of certain syzygies, and devoted his Ninth Memoir on Quantics (7th April, 1870) to the correction of his error and a further development of the theory in the light of Gordan's results.

The whole of this primal theory of invariants may now be regarded as a natural and elegant application of Lie's theory of continuous groups. The differential parameters, which in the ordinary theory of binary forms enable us to calculate new invariants from known ones, appear in a simple way as differential invariants of certain linear groups. The Lie theory may be illustrated by a simple example :

Consider the binary quadratic form

$$f \equiv a_0 x^2 + 2a_1 xy + a_2 y^2 .$$

Applying to f the linear transformation

$$(1) \qquad x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y',$$

we obtain the quadratic form

$$f' \equiv a'_0 x'^2 + 2a'_1 x'y' + a'_2 y'^2 ,$$

where the coefficients are readily found to be

$$(2) \qquad \begin{cases} a'_0 = \alpha^2 a_0 + 2\alpha\gamma a_1 + \gamma^2 a_2 , \\ a'_1 = \alpha\beta a_0 + (\alpha\delta + \beta\gamma) a_1 + \gamma\delta a_2 , \\ a'_2 = \beta^2 a_0 + 2\beta\delta a_1 + \delta^2 a_2 . \end{cases}$$

We may easily verify the following identity :

$$a'_0 a'_2 - a'^2_1 = (\alpha\delta - \beta\gamma)^2 (a_0 a_2 - a_1^2).$$

Hence $a_0 a_2 - a_1^2$ is an invariant of the form f. In the group theory it is an invariant of the group of linear homogeneous transformations (2) on the three parameters a_0, a_1, a_2.

The only covariant of f is known to be f itself. In the Lie theory it appears as the *invariant* of a linear homogeneous group on five variables. x, y, a_0, a_1, a_2, the transformations being defined by the equations (2) together with (1) when inverted.

In general, the invariants of a binary form of degree n are defined by a linear homogeneous group on its $n+1$ coefficients, its covariants by a group on $n+3$ variables.

As in all problems in continuous groups, the detailed developments are greatly simplified by employing the infinitesimal transformations of the groups

concerned. It is readily proven by the group theory that all invariants and covariants are expressible in terms of a finite number of them. This result is, however, not equivalent to the algebraic result that all rational integral invariants (including covariants) are expressible rationally and integrally in terms of a finite number of such invariants.

Twenty years ago, in my "Bibliography of Hyper-Space and Non-Euclidean Geometry," (American Journal of Mathematics, Vol. I., Nos. 2 and 3, 1878) I cited seven of Cayley's papers written before 1873 :

I. Chapters in the Analytical Geometry of (n) Dimensions. Camb. Math. Jour., Vol, IV., 1845, pp. 119-127.

II. Sixth Memoir on Quantics. Phil. Trans., vol. 149. pp. 61-90, (1859).

III. Note on Lobatchewsky's Imaginary Geometry. Phil. Mag. XXIX,. pp. 231-233, (1865).

IV. On the rational transformation between two spaces. Lond. Math. Soc Proc. III., pp. 127-180, (1869-71).

V. A Memoir on Abstract Geometry. Phil. Trans. CLX., pp. 51-63, (1870).

VI. On the superlines of a quadric surface in five dimentional space. Quarterly Journ., Vol. XII., pp. 176-180, (1871-72).

VII. On the Non-Euclidean Geometry. Clebsch. Math. Ann. V., pp. 630-634, (1872).

Four of these pertain to Hyper-Space, and in that Bibliography I quoted Cayley as to its geometry as follows :

"The science presents itself in two ways,—as a legitimate extention of the ordinary *two*- and *three*-dimensional geometries ; and as a need in these geometries and in analysis generally. In fact, whenever we are concerned with quantities connected together in any manner, and which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as coördinates of a point in a plane or in space : for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation : but this can only be obtained by means of the notion of a space of the proper dimensionality : and to use such a representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions : the conditions imply relations between the coefficients in the equation of the curve ; and for the better understanding of these relations it was expedient to consider the coefficients as the coördinates of of a point in a space of the proper dimensionality."

For a dozen years after it was written, the Sixth Memoir on Quantics would not have been enumerated in a Bibliography of non-Euclidean geometry, for its author did not see that it gave a generalization which was identifiable with that initiated by Bolyai and Lobachévski, though afterwards, in his Address to the British Association, 1883, he attributes the fundamental idea involved to

Riemann, whose paper was written in 1854. Says Cayley: "In regarding the physical space of our experience as possibly non-Euclidean, Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four dimensional space."

What the Sixth Memoir was meant to do was to base a generalized theory of metrical geometry on a generalized definition of distance. As Cayley himself says: " . . . the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connection with another figure, viz., the conic termed the absolute."

The fundamental idea that a metrical property could be looked at as a projective property of an extended system had occurred in the French school of geometers. Thus Laguerre (1853) so expresses an angle. Cayley generalized this French idea, expressing all metrical properties as projective relations to a fundamental configuration.

We may illustrate by tracing how Cayley arrives at his projective definition of distance. Two projective primal figures of the same kind of elements and both on the same bearer are called conjective. When in two conjective primal figures one particular element has the same mate to whichever figure it be regarded as belonging, then every element has this property. Two conjective figures such that the elements are mutually paired (coupled) form an involution. If two figures forming an involution have self correlated elements, these are called the double elements of the involution. An involution has at most two double elements; for were three self-correlated, all would be self-correlated. If an involution has two double elements these separate harmonically any two coupled elements. An involution is completely determined by two couples.

From all this it follows that two point-pairs A, B and A_1, B_1 define an involution whose double points D, D_1 are determined as that point-pair which is harmonically related to the two given point-pairs. Let the pair A, B be fixed and called the Absolute. Two new points A_1, B_1 are said (by definition) to be equidistant from a double point D defined as above. D is said to be the 'center' of the pair A_1, B_1. Inversely, if A_1 and D be given, B_1 is uniquely determined. Thus starting from two points P and P_1, we determine P_2 such that P_1 is the center of P and P_2, then determine P_3 so that P_2 is the center of P_1 and P_3, etc.; also in the opposite direction, we determine an analogous series of points P_{-1}, P_{-2}, ... We have therefore a series of points

$$\ldots\ldots, P_{-2}, P_{-1}, P, P_1, P_2, P_3, \ldots\ldots$$

at 'equal intervals of distance.' Taking the points P, P_1 to be indefinitely near to each other, the entire line will be divided into a series of equal infinitesimal elements.

In determining an analytic expression for the distance of the two points, Cayley introduced the inverse cosine of a certain function of the coördinates, but

in the Note which he added in the Collected Papers he recognizes the improvement gained by adopting Klein's assumed definition for the distance of any two points P, Q: dist. $(PQ)=c\log\frac{AP.BQ}{AQ.BP}$, where A, B are the two fixed points giving the Absolute.

This definition preserves the fundamental relation

$$\text{dist. } (PQ)+\text{dist. } (QR)=\text{dist. } (PR).$$

In this note (Col. Math. Papers, Vol. 2, p. 604) Cayley discusses the question whether the new definitions of distance depend upon that of distance in the ordinary sense, since it is obviously unsatisfactory to use one conception of distance in defining a more general conception of distance. His earlier view was to regard coördinates "not as distances or ratios of distances, but as an assumed fundamental notion, not requiring or admitting of explanation." Later he regarded them as "mere numerical values, attached arbitrarily to the point, in such wise that for any given point the ratio $x:y$ has a determinate numerical value," and inversely.

But in 1871 Klein had explicitly recognized this difficulty and indicated its solution. He says: "The cross ratios (the sole fixed elements of projective geometry) naturally must not here be defined, as ordinarily happens, as ratios of sects, since this would assume the knowledge of a measurement. In von Staudt's Beitraegen Zur Geometrie der Lage (§ 27, n. 393), however, the necessary materials are given for defining a cross ratio as a pure number. Then from cross ratios we may pass to homogeneous point—and plane—coördinates, which indeed are nothing else than the relative values of certain cross ratios, as von Staudt has likewise shown (Beitrage, § 29, n. 411)."

This solution was not satisfactory to Cayley, who did not think the difficulty removed by the observations of von Staudt that the cross ratio (A, B, P, Q) has "independently of any notion of distance the fundamental properties of a numerical magnitude, viz., any two such ratios have a sum and also a product, such sum and product being each of them a like ratio of four points determinable by purely descriptive construction."

Consider, for example, the product of the ratios (A, B, P, Q) and (A', B', P', Q'). We can construct a point R such that $(A', B', P', Q')=(A, B, Q, R)$. The product of (A, B, P, Q) and (A, B, Q, R) is said to be (A, B, P, R). This last definition of a product of two cross ratios, Cayley remarks, is in effect equivalent to the assumption of the relation

$$\text{dist. } (PQ)+\text{dist. } (QR)=\text{dist. } (PR).$$

The original importance of this memoir to Cayley lay entirely in its exhibiting metric as a branch of descriptive geometry. That this generalization of distance gave pangeometry was first pointed out by Klein in 1871. Klein showed that if Cayley's Absolute be real, we get Bolyai's system; if it be imagi-

nary, we get either spheric or a new system called by Klein single elliptic; if the Absolute be an imrginary point pair, we get parabolic geometry, and if, in particular, the point pair be the circular points, we get ordinary Euclid.

It is maintained by B. A. W. Russell in his powerful Essay on the Foundations of Geometry (Cambridge, 1897) "that the reduction of metrical to projective properties, even when, as in hyperbolic geometry, the Absolute is real, is only apparent, and has merely a technical validity."

Cayley first gave evidence of acquaintance with non-Euclidean geometry in 1865 in the paper in the Philosophical Magazine above mentioned. Though this is six years after the Sixth Memoir, and though another six was to elapse before the two were connected, yet this is, I think, the very first appreciation of Lobachévski in any mathematical journal. Baltzer has received deserved honor for in 1866 calling the attention of Hoüel to Lobachévski's 'Geometrische Untersuchungen,' and from the spring thus opened actually flowed the flood of ever broadening non-Euclidean research. But whether or not Cayley's path to these gold-fields was ever followed by anyone else, still he had therein marked out a claim for himself a whole year before the others.

In 1872 after the connection with the Sixth Memoir had been set up, Cayley takes up the matter in his paper in the Mathematische Annalen 'On the Non-Euclidean Geometry,' which begins as follows: "The theory of the Non-Euclidian Geometry as developed in Dr. Klein's paper, 'Ueber die Nicht-Euclidische Geometrie,' may be illustrated by showing how in such a system we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the 'hyperbolic' case of plane geometry; viz., the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to the points *within* the circle. I use the simple letters, a, A, . . to denote (linear or angular) distances measured in the ordinary manner; and the same letters with a superscript stroke $\bar{a}$, $\bar{A}$, . . to denote the same distances measured according to the theory. The radius of the absolute is for convenience taken to $=1$, the distance of any point from the center can therefore be represented as the sine of an angle.

The distance $\overline{BC}$, or say $\bar{a}$, of any two points B, C is by definition as follows:

$$\bar{a} = \tfrac{1}{2} \log \frac{BI.CJ}{BJ.CI}$$

(where I, J are the intersections of the line BC with the circle)."

As for the trigonometry, "the formulae are in fact similar to those of spherical trigonometry with only $\cosh \bar{a}$, $\sinh \bar{a}$, etc., instead of $\cos a$, $\sin a$, etc."

Cayley returned again to this matter in his celebrated Presidential Address to the British Association (1883), saying there: "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid

geometry." "But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows?"

The very next year this ever interesting subject recurs in the paper (May 27, 1884) "On the Non-Euclidean Plane Geometry." "Thus the geometry of the pseudo-sphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface; for in regard to the descriptive geometry, the statement requires some qualification . . . this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

I would remark that this realization of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable."

But here this necessarily brief notice must abruptly stop.

Cayley in addition to his wondrous originality was assuredly the most learned and erudite of mathematicians. Of him in his science it might be said, he knew everything, and he was the very last man who ever will know everything. His was a very gentle, sweet character. Sylvester told me he never saw him angry but once, and that was (both were practicing law!) when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an excess of disgust, Cayley dashed the documents upon the floor.

Austin, Texas, February, 1899.

NOTE ON SPHERICAL GEOMETRY.

By G. B. M. ZERR, A. M., Ph. D., Chester, Pa.

DEFINITION. Two arcs of great circles drawn from the vertex of a spherical triangle making equal angles with the spherical bisector of the angle at that vertex are called *isogonal conjugate arcs*. If three arcs drawn through the vertices of a spherical triangle are concurrent, their *isogonal conjugates* with respect to the angles at these vertices are also concurrent.

Let the arcs AM_a, BM_b, CM_c be concurrent at M. To prove that their isogonal conjugates AK_a, BK_b, CK_c are concurrent.

Fig. 1. Let $BM_a=a_1$, $CM_a=a_2$, $CM_b=b_1$, $AM_b=b_2$, $AM_c=c_1$, $BM_c=c_2$, $BK_a=a_3$, $CK_a=a_4$, $CK_b=b_3$, $AK_b=b_4$, $AK_c=c_3$, $BK_c=c_4$.

$\angle CMM_b=x$, $\angle CMM_a=y$, $\angle BMM_a=z$, $\angle CAM_a=\angle BAK_a=\theta$, $\angle BAM_a=$

Fig. 1.

$\angle CAK_a = A-\theta$, $\angle CBM_b = \angle ABK_b = \varphi$, $\angle ABM_b = \angle CBK_b = B-\varphi$, $\angle BCM_c = \angle ACK_c = \psi$, $\angle ACM_c = \angle BCK_c = C-\psi$.

Now we have

$$\sin a_1 : \sin BM = \sin z : \sin M_a$$
$$\sin a_2 : \sin CM = \sin y : \sin M_a.$$

$$\therefore \frac{\sin a_1}{\sin a_2} = \frac{\sin BM \sin z}{\sin CM \sin y} \quad \ldots\ldots (1).$$

Similarly, $$\frac{\sin b_1}{\sin b_2} = \frac{\sin CM \sin x}{\sin AM \sin z} \quad \ldots\ldots (2).$$

$$\frac{\sin c_1}{\sin c_2} = \frac{\sin AM \sin y}{\sin BM \sin x} \quad \ldots\ldots (3).$$

Multiplying (1), (2), (3) together we get as the condition of concurrence, the following:

$$\frac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = 1 \quad \ldots\ldots (4).$$

Now
$$\sin a_3 : \sin AK_a = \sin\theta : \sin B$$
$$\sin a_4 : \sin AK_a = \sin(A-\theta) : \sin C.$$

$$\therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin\theta \sin C}{\sin(A-\theta)\sin B}.$$

But
$$\sin a_1 : \sin AM_a = \sin(A-\theta) : \sin B$$
$$\sin a_2 : \sin AM_a = \sin\theta : \sin C.$$

$$\therefore \frac{\sin\theta}{\sin(A-\theta)} = \frac{\sin a_2 \sin C}{\sin a_1 \sin B}. \qquad \therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin a_2 \sin^2 C}{\sin a_1 \sin^2 B}.$$

Similarly, $$\frac{\sin b_3}{\sin b_4} = \frac{\sin b_2 \sin^2 A}{\sin b_1 \sin^2 C}, \qquad \frac{\sin c_3}{\sin c_4} = \frac{\sin c_2 \sin^2 B}{\sin c_1 \sin^2 A}.$$

$$\therefore \frac{\sin a_3 \sin b_3 \sin c_3}{\sin a_4 \sin b_4 \sin c_4} = \frac{\sin a_2 \sin b_2 \sin c_2}{\sin a_1 \sin b_1 \sin c_1} = 1 \quad \ldots\ldots (5).$$

$\therefore AK_a$, BK_b, CK_c are concurrent at K.

The two points M, K are called *spherical isogonal conjugate points* with respect to the triangle.

Let MD, ME, MF, KG, KH, KI be the spherical distances (perpendicular arcs) of M, K from the sides a, b, c of the triangle.

Then if $\sin MD : \sin ME : \sin MF = \beta : \gamma : \delta$, it can be demonstrated that $\sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$, as follows:

$$\sin MD = \sin CM \sin\psi,$$
$$\sin ME = \sin CM \sin(C-\psi).$$

$$\therefore \frac{\sin MD}{\sin ME} = \frac{\sin\psi}{\sin(C-\psi)} = \beta/\gamma. \quad \therefore \sin MD : \sin ME = \beta : \gamma.$$

$$\sin KG = \sin CK \sin(C-\psi),$$
$$\sin KH = \sin CK \sin\psi.$$

$$\therefore \frac{\sin KG}{\sin KH} = \frac{\sin(C-\psi)}{\sin\psi} = \gamma/\beta. \quad \therefore \sin KG : \sin KH = \gamma : \beta = 1/\beta : 1/\gamma.$$

In the same way it can be shown that $\sin KH : \sin KI = 1/\gamma : 1/\delta$.

$\therefore \sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$.

DEFINITION. If two arcs of great circles are drawn from the vertex of a spherical triangle cutting the base equally distant from the mid-point, the two arcs thus drawn are called *isotomic conjugate arcs.*

If any three arcs drawn from the vertices A, B, C of a spherical triangle to the opposite sides are concurrent, their *isotomic* conjugates are also concurrent.

This follows at once from (5) since $a_2 = a_3$, $a_1 = a_4$, $b_2 = b_3$, $b_1 = b_4$, $c_2 = c_3$, $c_1 = c_4$.

The two points thus determined are called *spherical isotomic conjugate points.*

Fig. 2. Let RD, RE, RF, PG, PH, PI be the spherical distances of R, P from the sides a, b, c of the triangle.

Then if $\sin RD : \sin RE : \sin RF = l : m : r$ it can be demonstrated that $\sin PG : \sin PH : \sin PI = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}$ as follows:

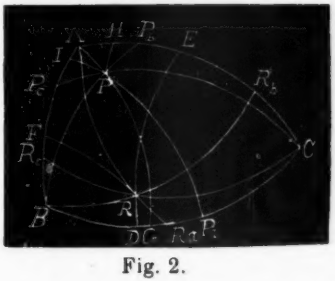

Fig. 2.

Let $BR_a = CP_a = a_1$, $BP_a = CR_a = a_2$,
$CR_b = AP_b = b_1$, $CP_b = AR_b = b_2$,
$BR_c = AP_c = c_1$, $BP_c = AR_c = c_2$.

Then $\sin RD = \sin RC \sin RCD$,
$\sin RE = \sin RC \sin RCE$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin RCD}{\sin RCE}.$$

But
$$\sin RCD : \sin R_c = \sin c_1 : \sin a$$
$$\sin RCE : \sin R_c = \sin c_2 : \sin b.$$

$$\therefore \frac{\sin RCD}{\sin RCE} = \frac{\sin b \sin c_1}{\sin a \sin c_2}. \quad \therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin c_1}{\sin a \sin c_2} = l/m.$$

$$\sin PG = \sin PC \sin PCG$$
$$\sin PH = \sin PC \sin PCH.$$

$$\therefore \frac{\sin PG}{\sin PH} = \frac{\sin PCG}{\sin PCH}.$$

But $\sin PCG : \sin P_c = \sin c_2 : \sin a$

$\sin PCH : \sin P_c = \sin c_1 : \sin b.$

$$\therefore \frac{\sin PCG}{\sin PCH} = \frac{\sin b \sin c_2}{\sin a \sin c_1}. \quad \therefore \frac{\sin PG}{\sin PH} = \frac{\sin b \sin c_2}{\sin a \sin c_1} = \frac{\sin^2 b}{\sin^2 a}.m/l.$$

$$\therefore \sin PG : \sin PH = \sin^2 b.m : \sin^2 a.l \text{ or } \sin PG : \sin PH = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b}.$$

In the same way it can be demonstrated that

$$\sin PH : \sin PI = \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

EXAMPLES. If M is the median point, K is the symmedian point.

In this case $\frac{\sin (C-\psi)}{\sin \psi} = \frac{\sin A}{\sin B}$.

$\therefore \sin MD : \sin ME : \sin MF = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C$.

$\sin KG : \sin KH : \sin KI = \sin A : \sin B : \sin C$.

If R is the point of concurrence of arcs drawn from the angles to the points of contact of the incircle with the opposite sides; P, its isotomic conjugate point, is the point of concurrence of arcs drawn from the angles to the points of contact of the ex-circles with the opposite sides.

In this case $c_1 = (s-b)$, $c_2 = (s-a)$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin (s-b)}{\sin a \sin (s-a)} = \frac{\cos^2 \frac{1}{2}B}{\cos^2 \frac{1}{2}A} = \frac{\sec^2 \frac{1}{2}A}{\sec^2 \frac{1}{2}B}.$$

$$\therefore \sin RD : \sin RE : \sin RF = \sec^2 \tfrac{1}{2}A : \sec^2 \tfrac{1}{2}B : \sec^2 \tfrac{1}{2}C.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{\cos^2 \frac{1}{2}A}{\sin^2 a} : \frac{\cos^2 \frac{1}{2}B}{\sin^2 b} : \frac{\cos^2 \frac{1}{2}C}{\sin^2 c}.$$

If we start with the in-center, whose distances from the sides are $1:1:1$, and take its isotomic conjugate we get a point whose distances from the sides are

$$\frac{1}{\sin^2 a} : \frac{1}{\sin^2 b} : \frac{1}{\sin^2 c}.$$

The isogonal conjugate of this last point is a point whose distances from the sides are $\sin^2 a : \sin^2 b : \sin^2 c$. By repeating this process we get a series of points whose distances from the sides are as $\sin^m a : \sin^m b : \sin^m c$, where m is an even positive or negative integer. For the symmedian point we have $\sin A : \sin B : \sin C = \sin a : \sin b : \sin c$. Starting with this point and alternating as above, we get a series of points whose distances from sides are as $\sin^n a : \sin^n b : \sin^n c$ where n is any odd positive or negative integer.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from February Number.]

XCI. Fig. 38 and Fig. 39.

The triangle is ABC. The construction in the two figures is evident. The Hindu Bhaskara, the author of this method, complimented his readers by condensing his proof into the single word, "Behold." We follow his example.

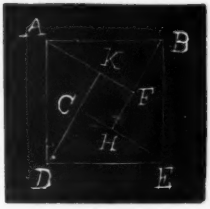

Fig. 38.

NOTE. The above is a conjectured proof of Pythagoras. See pages 50 and 123 of Cajori's "History of Elementary Mathematics."

XCII. Fig. 38.

$$AB^2 = 2AC.BC + CH^2$$
$$= 2AC.BC + (AH - AC)^2$$
$$= 2AC.BC + (BC - AC)^2$$
$$= BC^2 + AC^2.$$

Q. E. E.

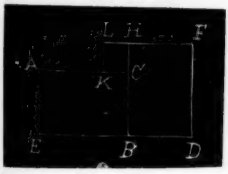

Fig. 39.

XCIII. Fig. 38.

Suppose BC is produced to meet AD, as at L.

Then let LBA be the given triangle, right-angled at A.

Now, the the area of the square on AB=the sum of the four triangles ABC, BEK, DEF and ADH, and the square $CHFK$; or, $AB^2 = 2AC.BC + CH^2$.

Again, $AC = (AL.AB) \div BL$, $BC = AB^2 \div BL$, and $CH = AH - AC = BC - AC$.

$$\therefore 2AC.BC + CH^2 = BC^2 + AC^2 = \frac{AB^4}{BL^2} + \frac{AL^2.AB^2}{BL^2}.$$

$$\therefore BL^2 = AB^2 + AL^2.$$

Q. E. D.

XCIV. Fig. 40.

$$AB^2 = CF^2 - 2AC.BC$$
$$= (BF+BC)^2 - 2AC.BC$$
$$= (AC+BC)^2 - 2AC.BC$$
$$= AC^2 + BC^2. \qquad \text{Q. E. D.}$$

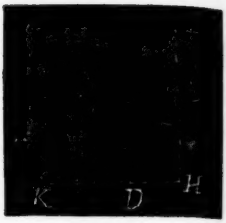

Fig. 40.

NOTE. If we join AE in Fig. 40, and treat the trapezoid $AEFC$ after the manner of the above demonstration, we shall have a slightly different proof. This is known as General Garfield's proof. The writers are indebted, for its reproduction, to Professor Coleman Bancroft, of Hiram College, who gives also the interesting information that the proof was the result of one of Mr. Garfield's *mathematical recreations* while in Congress.

XCV. Fig. 41. Make $BE=BC$, and draw DE perpendicular to AB.

Then $DE=DC$.

Now, the area of $ABC = \frac{1}{2}AC.BC = DE.BC + \frac{1}{2}AE.DE$.

But $DE = (BC.AE) \div AC$, and $AE = AB - BC$.

$$\therefore \tfrac{1}{2}AC.BC = \frac{BC^2(AB-BC)}{AC} + \tfrac{1}{2}\frac{(AB-BC)^2BC}{AC}$$

$$\therefore AC^2 = 2AB.BC - 2BC^2 + AB^2 - 2AB.BC + BC^2.$$

$$\therefore AB^2 = AC^2 + BC^2. \qquad \text{Q. E. D.}$$

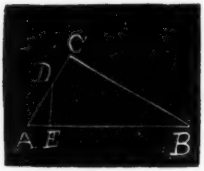

Fig. 41.

XCVI. Fig. 41.

Let ADE be the triangle right angled at E. Produce AD to C making $DC = DE$. Produce AE till it meets the perpendicular BC drawn from C.

Then are the two triangles similar.

Now, the area of $ADE = \frac{1}{2}AE.DE = \frac{1}{2}AC.BC - DC.BC$

$$= \tfrac{1}{2}\frac{DE(AD+DE)^2}{AE} - \frac{DE^2(AD+DE)}{AE}.$$

Whence, $AD^2 = AE^2 + DE^2$. Q. E. D.

XCVII. Fig. 42.

Let ABC be the triangle, right-angled at C. Draw the escribed circle O, tangent to the hypotenuse.

Designate the sides of the triangle by a, b, c, and the radius of the circle by r.

The area of the square $CPOS = r^2 = \frac{1}{2}ab + rc$.

But $r = \frac{1}{2}(a+b+c)$.

$\therefore$ By substitution, $\frac{1}{4}(a+b+c)^2 = \frac{1}{2}ab + \frac{1}{2}c(a+b+c)$.

$\therefore c^2 = a^2 + b^2$. Q. E. D.

Fig. 42.

IV. QUATERNION PROOFS.

XCVIII. Fig. 43.

Represent the sides as indicated in the figure.

Then $\gamma = \alpha + \beta$. Squaring, $\gamma^2 = \alpha^2 + 2S\alpha\beta + \beta^2$.

But, since angle C is right, $2S\alpha\beta = 0$.

$\therefore \gamma^2 = \alpha^2 + \beta^2$, or as lengths simply, changing signs, $AB^2 = AC^2 + BC^2$. Q. E. D.

Fig. 43.

From Hardy's *Elements of Quaternions*.

XCIX. Fig. 44.

Let ABC be a triangle, right-angled at C. Draw CD perpendicular to CB and equal to AC. Draw DB, which equals AB.

Represent vectors AC and CD by α, and vector CB by β.

$\therefore$ As vectors, $AB=AC+CB=\alpha+\beta$, and $DB=DC+CB=-\alpha+\beta$.

Squaring and adding, we have $AB^2+BD^2=2\alpha^2+2\beta^2$. For the corresponding *lines*, we have $AB^2+BD^2=2AC^2+2CB^2$.

$\therefore AB^2=AC^2+CB^2$. Q. E. D.

Fig. 44.

C. Fig. 45.

Let ABC be a triangle, right-angled at C. Complete the parallelogram $ADBC$. Draw the diagonal CD.

Fig. 45.

Represent vectors CB and AD by α, and vectors CA and BD by β.

$\therefore$ As vectors, $CD=CA+AD=\alpha+\beta$, and $AB=AD+DB=\alpha-\beta$.

Squaring and adding, we have $CD^2+BA^2=2\alpha^2+2\beta^2$. For the corresponding *lines*, we have $CD^2+BA^2=2CA^2+2CB^2$.

$\therefore 2BA^2=2CA^2+2CB^2$.

$\therefore AB^2=AC^2+CB^2$. Q. E. D.

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HISTORICAL NOTE.

BY FLORIAN CAJORI, PH. D., PROFESSOR OF PHYSICS, COLORADO COLLEGE, AUTHOR OF TWO HISTORIES OF MATHEMATICS.

[Contributed by the request of the authors.]

If the lengths of the sides of a triangle are, respectively, 3, 4, 5 units, then the figure is a right triangle. This fact was known to the early Egyptians, who, it appears, based upon it a method of laying out their temples. They determined a N. and S. line by accurate astronomical observation, then ran a line at right angles to this by means of a rope stretched around three pegs in such a way that the three sides of a triangle thus formed were to each other as 3 : 4 : 5, one of the legs of the right triangle being made to coincide with the N. and S. line.* Essentially the same process was described later by Heron of Alexandria, by the Hindu astronomers, and by Chinese writers. The Hindus took for the lengths of the sides 15, 36, 39, respectively. There is reason to believe that the Egyptian "rope-stretchers" existed as early as the time of King Amenemhat I., about 2300 B. C. If this date is correct, then this method of laying out right angles in the field by rope-stretching was in vogue fully 3000 years!

The discovery of the well-known property of the right triangle is ascribed by Greek writers to Pythagoras. The truth of the theorem for the special case when the sides are 3, 4, 5, respectively, he may have learned from the Egyptians. That the importance and beauty of this theorem of three squares was thoroughly appreciated by the Greeks is evident from the legend to which its discovery gave rise. Pythagoras is said to have been so jubilant over his great achievement, that he offered a hecatomb to the muses who inspired him. As the Pythagoreans believed in the transmigration of the soul and, for that reason, opposed the shedding of blood, the sacrifice was replaced in the traditions of the Neo-Pythagoreans by that of "an ox made of flour"! The proof given by Pythagoras for this theorem has not been handed down to us. That in *Euclid* I, 47 is due to Euclid himself. Much ingenuity has been expended in conjecture as to the nature of the proof given by Pythagoras. Some critics believe that the proof involved the consideration of special cases; that it was essentially that for the isosceles right triangle outlined by Plato in *Meno*,† in which a square is divided into isosceles right triangles. Other critics surmise that the Pythagorean proof was substantially the same as that given by the Hindu astronomer Bhaskara (about 1150 A. D.), who draws the right triangle four times in the square upon its hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging the small square and the four triangles in a different way, they can be shown, together, to make up the sum of the squares of the two sides. In another place Bhaskara gives a second

*M. Cantor, *Vorlesungen ueber Geschichte der Mathematik*, Vol. I, 1894, page 64.
†Cantor, *op. cit.* page 205.

demonstration of this theorem by drawing from the vertex of the right triangle a perpendicular to the hypotenuse and then suitably manipulating the proportions yielded by the similar triangles. This proof was unknown in Europe until it was rediscovered by the English mathematician, John Wallis.

Among Arabic authors the earliest proof, for the case of the isosceles right triangle, was giveu by Alchwarizmî, who lived in the early part of the 9th century. It is the same as that in Plato's *Meno*. The Persian mathematician, Nasîr Eddîn, who flourished during the early part of the 13th century, gave a new proof, which required the consideration of eight special cases.* Until six years ago this proof was attributed to more recent writers.

The theorem of Pythagoras has received several nicknames. In European universities of the Middle Ages it was called "magister matheseos," because examinations for the degree of A. M. (when held at all) appear usually not to have extended beyond this theorem, which, with its converse, is the last in the first book of Euclid. The name, "pons asinorum," has sometimes been applied to it, though usually this is the sobriquet for *Euclid*, I., 5. Some Arabic writers, Behâ Eddîn for instance, call the Pythagorean theorem, "figure of the bride." Curiously enough, this romantic appellation appears to have originated from a mistranslation of the Greek word $\nu\acute{\upsilon}\mu\phi\eta$, applied to the theorem by a Byzantine writer of the 13th century. This Greek word admits of two meanings, "bride" and "winged insect." The figure of the right triangle with the three squares suggests an insect, but Behâ Eddîn apparently translated the word as "bride."†

*See H. Suter in *Bibliotheca Mathematica*, 1892, pages 3 and 4.
†See P. Tannery in *L'Intermediare des Mathematiciens*, 1894, Vol. I, page 254.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

106. Proposed by ELMER SCHUYLER, High Bridge, N. J.

What is the amount of $1000 at compound interest for three years, at 6%, if it be compounded every instant?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and J. OWEN MAHONEY, B. E., M. Sc., Instructor in Mathematics, Carthage High School, Carthage, Texas.

Let A=amount, P=principal, r=rate, n=number of years, q=number of times interest is payable a year.

Then $A=P[1+(r/q)]^{qn}$. Let $q=rx$.

$\therefore A=P[1+(1/x)]^{nrx}=P\{[1+(1/x)]^{x}\}^{rn}=Pe^{rn}$ when x is infinite.

$\therefore A=1000e^{.18}=1000\times 1.19705=\$1197.05$.

II. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; J. D. CRAIG, Frankfort, Ky.; and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let A be the amount of P dollars for n years at r per cent. payable q times a year. Then $A=P[1+(r/q)]^{nq}=$

$$P\left[1+nr+\frac{n^2r^2}{1.2}-\frac{nr^2}{1.2.q}+\frac{n^3r^3}{1.2.3}+\frac{2n}{1.2.3q^2}-\frac{3n}{1.2.3.q}+\frac{n^4r^4}{1.2.3.4}+\ldots.\right].$$

Now if interest is to be compounded every instant, q is infinite and hence all terms in this series containing q will vanish, and we have

$$A=P\left[1+nr+\frac{n^2r^2}{1.2}+\frac{n^3r^3}{1.2.3}+\frac{n^4r^4}{1.2.3.4}+\ldots.\right]=Pe^{nr}.$$

$\therefore A=\$1000(2.71828)^{.18}=\$1197.462+.$

III. Solution by D. G. DORRANCE, Jr., Camden, N. Y.

The formula for the amount of (a) dollars for (n) years at $r\%$ interest compounded every xth part of a year is

$$a\left(1+\frac{r}{x}\right)^{nx}$$

which expanded by the Binomial Theorem becomes

$$a\left(1+nx\frac{r}{x}+\frac{nx(nx-1)}{1.2}\frac{r^2}{x^2}+\frac{nx(nx-1)(nx-2)}{1.2.3}\frac{r^3}{x^3}+\text{ etc.}\right)$$

which, when x is made infinitely large, becomes

$$a\left(1+nr+\frac{n^2r^2}{1.2}+\frac{n^3r^3}{1.2.3}+\frac{n^4r^4}{1.2.3.4}+\frac{n^5r^5}{1.2.3.4.5}+\text{ etc.}\right).$$

Make $a=\$1000.$, $n=3$, and $r=.06$, and the above becomes

$\$1000(1+.18+.0162+.000972+.00004374+.00000157464+\text{etc.})$
$=\$1000(1.19721731464+)=\$1197.21731464+$, the required amount.

Also solved by *CHAS. C. CROSS*, and *ALOIS F. KOVARIK*.

ALGEBRA.

91. Proposed by NELSON S. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.

Solve the following without making use of the determinant notation, and prove that the results obtained are the roots.

$$10x-2y+4z=5,$$
$$3x+5y-3z=7,$$
$$x+3y-2z=2.$$

I. Solution by H. C. **WHITAKER**, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Eliminate z: $12x+4y=9$.....(1),

$42x+14y=43$.....(2).

Multiplying (1) by 7, and (2) by 2,

$84x+28y=63$,

$84x+28y=86$.

Whence the values of x, y, and z are infinite.

Multiply (1) by 43, and (2) by 9; equate results and reduce:

$y=-3x$.

Similarly, $z=-4x$.

The values $a/0$, $-(3a/0)$, $-(4a/0)$ prove in all three equations.

G. B. M. ZERR, *J. K. ELLWOOD*, *M. A. GRUBER*, and *ELMER SCHUYLER* each show that the equations are not simultaneous.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Combining (1) and (3) we find $3x+y=\frac{9}{4}$.....(4).

Combining (1) and (2) we find $3x+y=\frac{43}{14}$.....(5).

Combining (2) and (3) we find $3x+y=8$.....(6).

These three results are incompatible.

From (1)+(3) we get $15x-19y+18z=0$, or $15(x/z)-19(y/z)=-18$.

From (1)+(3)−(2) we get $8x-4y+5z=0$, or $8(x/z)-4(y/z)=-5$.

Solving these we find the ratios $x/z=-\frac{1}{4}$, $y/z=\frac{3}{4}$.....(7).

The only answers to (4), (5), or (6) are $x=-\infty$, $y=+\infty$.

Similarly, $z=+\infty$, and the ratio of these infinities is given from (7): $x:y:z :: -1:3:4$.

Writing (1), $10(x/z)-2(y/z)+4=5/z=0$, since $z=\infty$.

$\therefore 10(-\frac{1}{4})-2(\frac{3}{4})=(-16/4)=-4$, which proves; similarly for (2) and (3).

III. Solution by CHARLES C. CROSS, Libertytown, Md.

Take the general equations,

$$ax+by+cz=d,$$
$$a'x+b'y+c'z=d',$$
$$a''x+b''y+c''z=d''.$$

From which,

$$x=\frac{b'c''d+bc'd''+b''cd'-b'cd''-bc''d'-b''c'd}{ab'c''+a''bc'+a'b''c-a''b'c-a'bc''-ab''c'},$$

$$y=\frac{ac''d'+a'cd''+a''c'd-a''d'c-a'c''d-ad''c'}{D},$$

$$z=\frac{ab'd''+a'b''d+a''bd'-a''b'd-a'bd''-ab''d'}{D}.$$

Whence by substitution $x=\frac{2}{0}$, $y=-\frac{6}{0}$, and $z=-\frac{9}{0}$.

From this we see that a problem is impossible when

$$ab'c''+a''bc'+a'b''c-a''b'c-a'bc''-ab''c'=0.$$

IV. Solution by J. M. BOORMAN, Woodmere, Long Island, N. Y.

(1), (2), or (3) requires a *modulus*, otherwise it lacks one congruous root for some value of x, y, or z. If (1) and (3) be held congruous, modulus (2)$=-2.875$. $\therefore$ Constant (2)$=7+M=4.125$.

$\therefore$ $x=1.3,\ 1.4,$ etc.; $2,\ 2.1\ldots\ldots$
$y=-1.65,\ -1.95,$ etc.; $-3.75,\ -4.05\ldots\ldots$
$z=-2.825,\ 3.225,$ etc.; $-5.625,\ -6.925\ldots\ldots$
} The law is obvious. $M=-2.875$ ....(A).

The above, with modulus (1)$=23$ taking (2) and (3) congruent, modulus (3)$=1.6\frac{3}{7}$. (1) and (2) taken congruent and the infinite series (B) and (C) are the required solution and roots.

PROOF. $\frac{2}{3}(2)$ is $2x+3\frac{1}{3}y=4\frac{2}{3}+2z$; $\frac{1}{3}(1)$ is $5x-y=2.5-2z$. Subtracting (3) from the first of the above and adding it to the second, we have

$$x+\tfrac{1}{3}y=2.66\tfrac{2}{3} \text{ and } 6x+2y=4.5\ldots\ldots(4).$$
$$x+\tfrac{1}{3}y=0.75 \quad \text{or} \quad x+\tfrac{1}{3}y=0.75.$$

Defect of $2z$ in (2), $-1.91\frac{2}{3}$ of $3z$=modulus (2)$=-2.875$.

If $10x$, $-2y$, etc.$=$; ..... $\therefore$ $y=-3.75$; $z=-5.625$ as above.
(1)$=20+7.5-22.5=5$.
(2)$=3x=6-18.75+16.875=7-M=4.125$.
(3) $x=2-11.25+11.25=2$.
Compute for $x=2.1$ for series (A) and its law.
Similarly, take (2) (3) congruent. $\therefore$ Modulus (1)$=23$. $\therefore$ (B).
Similarly, take (1) (2) congruent. $\therefore$ Modulus (3)$=1.6\frac{3}{7}$. $\therefore$ (C).
$\therefore$ $x_1=2,\ 2.2,\ 2.4,\ 2.6,\ 2.8,$ etc. Modulus (1)$=23$.
$y_1=2.\ 1.4.\ 0.8,\ 0.2,\ -0.4,$ etc. Constant (1)$=5+23=28$.
$z_1=3,\ 2.2,\ 1.4,\ 0.8,\ -0.2,$ etc. .....(B).
$x_2=2,\ 2.1.\ 2.2,$ etc.
$y_2=-2.9\frac{2}{7},\ -3.2\frac{2}{7},\ -3.5\frac{2}{7},$ etc. Modulus (3)$=1.6\frac{3}{7}$.
$z_2=-5.2\frac{1}{7},\ -5.6\frac{1}{7},\ 6.0\frac{1}{7},$ etc. Constant (3)$=36\frac{3}{7}$.
} ..... (C).
Laws of (B) and (C) are obvious.

[NOTE on Problem 89. I have been interested with solutions of the problem "Solve $\left\{\begin{matrix} x^2+y=7 \\ x+y^2=11 \end{matrix}\right.$ by quadratics," the eleventh solution of which appeared in the February number of the MONTHLY.

This example was given me to solve about forty-three years ago, and hardly a year has passed since without having been requested to solve it. A so called quadratic solution appeared in Henkle's "Notes and Queries" about twenty-five years ago; and Professor Schuyler inserted substantially the same solution in the edition of his Algebra of 1881. Several solutions appeared in the mathematical columns of the *New England Journal of Education* about fifteen years ago.

All the solutions in which the values $x=2$ and $y=3$ are found by completing a square are *tentative* and depend indirectly upon the values sought. Take, for example, V. (*b*). $x^2+y=7$ is written $y-3=4-x^2$ and $x+y^2=11$ is written $y^2-9=2-x$. That is, first, y minus the value of y is equal to the square of the value of x minus the square of x; and second, y^2 minus the square of the value of y is equal to the value of x

minus x; and the ability to get an equation of which the square can be completed depends upon this operation,

In the seventh solution Dr. Zerr lets $t=3-y$ and gets $x^2-4=t$ and $x-2=t(3-y)$. Now the only integral value of t that will satisfy this assumption is zero; and this gives $x=2$ and $y=3$. And the the rest of the work is only necessary to show a little algebraic skill in getting a form the square of which can be completed. We are to observe, too, that when the square root is taken the double sign must be excluded.

In regard to the eleventh solution I ask how the solver got the $10ab=40+10b$ to subtract from (5)? I think he will admit that he found it by trial; or if not, he knew the values sought and from them found the member by which (4) must be multiplied to get this result. In every similar example this number is $2(y^2-x^2)(y>x)$. In the present example $2(9-4)=10$. Had the example been $\left.\begin{matrix} x^2+y=9 \\ x+y^2=27 \end{matrix}\right\}$ when $x=2$ and $y=5$, the number with which (4) must be multiplied is $2(25-4)=42$. So by knowing the roots in this mode of solution we always can find what to subtract to complete the square. But if we do not know the roots sought the finding of this quantity is a mere work *trial*; and would require much more labor than to find the values by inspection.

So I have come to the conclusion that when there are integral values for x and y they are best found by inspection, and the other values are best found by the use of Sturm's Theorem and Horner's Method of approximation.

M. C. Stevens.

Purdue University.

This problem, with four different solutions (all of which have been published in the Monthly) may be found in the *School Visitor*, Vol. III., pp. 114-115, and the editor, Mr. John S. Royer, calls it the "Yale Problem." Three solutions of this problem may also be found in *The Mathematical Visitor*, Vol. II., p. 3, and two solutions of it appeared in Vol. II., p. 25, of *The Analyst*.

$x^2+y=a$ is the equation of a parabola whose axis coincides with the Y-axis of reference, and its infinite branches extending in the negative direction; $x+y^2=b$ is the equation of a parabola whose axis coincides with the X-axis of reference and its infinite branches extending in the negative direction. These two curves may intersect in four points, intersect in two points and touch in one, intersect in two points, touch in one point or not intersect at all. The equations considered as simultaneous may, therefore, have four real roots, all different; four real roots, two equal and two different; two real roots, equal or different, and two imaginary roots; or four imaginary roots. The solution of the general case leads to a biquadratic which cannot be solved by quadratics.

Editor F.]

GEOMETRY.

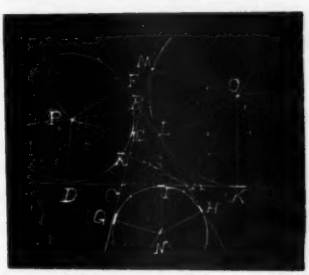
108. Proposed by NELSON L. RORAY, Bridgeton, N. J.

ABC is a triangle. O_1, O_2, O_3 centers of escribed circles. Prove altitudes of triangle $O_1O_2O_3$ are concurrent at center of incribed circle.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take the figure of my solution of Problem 97, Vol. V, No. 10, page 231, The American Mathematical Monthly.

We are to prove BN, AP, and CO the respective altitudes of $\triangle PON$, and passing through S, the center of inscribed circle of $\triangle ABC$.



$\triangle BGN$ and $\triangle BHN$ are equal; for $NG=NH$ (radii of same circle); $\angle BGN=\angle BHN$ =right triangle (radius to point of tangency), and BN is common.

$\therefore \angle ABN=\angle CBN$; and as BN bisects $\angle ABC$, BN passes through S, the center of inscribed circle.

$\angle ABO+\angle ABN+\angle CBN+\angle CBP=$ two right triangles. But $\angle ABO=\angle CBP$ (property of escribed circles), and $\angle ABN=\angle CBN$. $\therefore 2\angle ABO=2\angle ABN=$two right angles. Whence $\angle ABO+\angle ABN=\angle NBO=$right triangle.

$\therefore$ BN is perpendicular to PO.

$\therefore$ BN is the altitude of $\triangle PON$ let fall from the vertex N, and passes through the center of the inscribed circle of $\triangle ABC$.

In a similar manner, AP and CO can be proved the other two altitudes passing through S.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

ABC is the pedal triangle of $O_1O_2O_3$.

The altitudes of $O_1O_2O_3$ bisect the angles of its pedal triangle.

$\therefore$ The altitudes are concurrent at the in-center of ABC.

III. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Using trilinears, and ABC as the triangle of reference, the equation to BC, CA, AB being $\alpha=0$.....(1), $\beta=0$.....(2), $\gamma=0$.....(3), respectively, and noticing that O_1O_2, O_2O_3, O_3O_1 are the external bisectors of angles C, A, B, respectively, we have their equations, in order,

$$\alpha+\beta=0..... (4),\ \beta+\gamma=0.....(5),\ \gamma+\alpha=0.....(6).$$

Any line through the intersection of (4) and (5) is given by $\alpha+\beta+k(\beta+\gamma)=0$, or $(1+k)\alpha+\beta+k\gamma=0$.....(7).

The condition that (7) is perpendicular to (6) is

$$(1+k)(1-\cos A-\cos B-\cos C)=0.....(8).$$

This in general requires that $k=-1$.....(9), and (7) is $\beta-\gamma=0$..... (10), the internal bisector of $\angle C$. This really proves the theorem.

IV. Solution by P. H. PHILBRICK, C. E., Lake Charles, La.; G. I. HOPKINS, A. M., Professor of Mathematics and Physics, High School, Manchester, N. H.; W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.; MELVIN ENGER, Decorah, Ia.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Tex.

In the figure of the first solution, change N to O_2, P to O_1, O to O_3, and S to O.

Since COO_3 bisects the angle C, it passes through the centers O of the inscribed circle and O_3 of the escribed circle. Moreover, since CO_1 bisects the supplement of C, O_1CO_3 is a right angle. Hence CO_3 is an altitude of $O_1O_2O_3$. For the same reason, AO_1 as well as BO_2 pass through O and are altitudes of $O_1O_2O_3$.

Excellent demonstrations of this proposition were also furnished by *COOPER D. SCHMITT*, *E. D. SCALES*, *J. SCHEFFER*, and *CHAS. C. CROSS*. Mr. Cross sent in two different demonstrations.

CALCULUS.

83. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

From a given point, P, in the base, AB, of a triangle, to inscribe in the latter the minimum triangle, if its angle at P is given.

I. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

Let PQR be the triangle whose area is to be a minimum. Denote the segments into which P divides the base by a and b and the angles opposite them by θ and φ. Then since the angle PQR is constant $\theta+\varphi=$constant.

Now $PQ=\frac{a\sin A}{\sin\theta}$, $PR=\frac{b\sin B}{\sin\varphi}$.

In order that the triangle may be a minimum it is evident therefore that $\sin\theta\sin\varphi$ must be a *maximum*.

$\therefore \sin\varphi\cos\theta-\sin\theta\cos\varphi=0$.

$\therefore \sin(\varphi-\theta)=0$. $\therefore \varphi=\theta$. Or if it be required to *construct* the triangle it will be found that $\angle RPB=A+\frac{1}{2}(C-\psi)$ where ψ is the given angle at P.

II. Solution by the PROPOSER.

Let PQR be an inscribed triangle, Q lying in AC and R in BC. Denote the given angle QPR by β, and the given distances PA and PB respectively, by m and n. We have

$$QP=\frac{m\sin A}{\sin(A+\theta)},\quad RP=\frac{n\sin B}{\sin(\beta-B+\theta)}.$$

Since the area of $\triangle APQ$, $\angle QPR$ being constant, depends upon the product of AP and RP, the area is a minimum if $\sin(A+\theta)\sin(\beta-B+\theta)$ is a maximum. Putting this product$=M$, we find

$$\frac{\partial M}{\partial\theta}=\sin(A-B+\beta+2\theta),\text{ and }\frac{\partial^2 M}{\partial\theta^2}=2\cos(A-B+\beta+2\theta).$$

From $\sin(A-B+\beta+2\theta)=0$, we obtain $A-B+\beta+2\theta=180$, since $A-B+\beta+2\theta=0$, would make $\partial^2M/\partial\theta^2$ a positive quantity, and furnish a minimum instead of a maximum.

$\therefore \theta=90^\circ-\frac{1}{2}(A-B+\beta)$.

If $\beta=C$, $\theta=B$, and in this case the minimum triangle would have its sides QP and RP parallel to BC and AC respectively.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ELMER SCHUYLER, High Bridge, N. J.

$\angle APQ=\theta$, $\angle QPR=\lambda$.

$\therefore PQ=d\sin A\operatorname{cosec}(A+\theta)$, $PR=(c-d)\sin B\operatorname{cosec}(\lambda+\theta-B)$.

$\therefore \operatorname{cosec}(A+\theta)\operatorname{cosec}(\lambda+\theta-B)=$minimum.

This is the case when $\cot(A+\theta)+\cot(\lambda+\theta-B)=0$.

$\therefore\ \theta=\frac{1}{2}\pi-\frac{1}{2}(A+\lambda-B)$.

$\therefore\ PQ=d\sin A\operatorname{cosec}\frac{1}{2}(\lambda+C)$.

$PR=(c-d)\sin B\operatorname{cosec}\frac{1}{2}(\lambda+C$.

Geometrical construction : Upon AB describe a segment containing an angle $=\angle C+\angle\lambda$. At P erect PO perpendicular to AB meeting the circle in O ; draw OR, OQ perpendicular to BC, AC, respectively. Then PQR is the minimum triangle required.

IV. Solution by P. H. PHILBRICK, C.E., Chief Engineer, Kansas City, Watkins & Gulf Railway, Lake Charles, La.

Let the figure represent the triangle. Let angle at $P=2\theta$. Drop the perpendicular PD upon the nearest side, then draw Pa and Pa' making angles DPa and DPa' each equal to θ. Paa' is the triangle required.

To prove that this triangle is less than any other inscribed triangle having the same vertex angle at P, draw Pb and Pb' making angles aPb and $a'Pb'$ each $=x$.

Let $PD=p$. Now $aa'=2p\tan\theta$.

Also $bb'=bD+Db'=p[\tan(\theta-x)+\tan(\theta+x)]$.

$$\tan(\theta-x)=\frac{\tan\theta-\tan x}{1+\tan\theta\tan x}\text{ and }\tan(\theta+x)=\frac{\tan\theta+\tan x}{1-\tan\theta\tan x}.$$

Substituting and reducing, we find,

$$bb'=2p\tan\theta\times\frac{1+\tan^2 x}{1-\tan^2\theta\tan^2 x}>2p\tan\theta.$$

$\therefore\ bb'>aa'$ and triangle $Pbb'>$triangle Paa'.

Second Proof.

$$\tan(\theta-x)=(\theta-x)+\frac{(\theta-x)^3}{3}+\ldots\ldots,\text{ and }\tan(\theta+x)=(\theta+x)+\frac{(\theta+x)^3}{3}+\ \ldots\ldots;$$

$$\text{also } 2\tan\theta=2\theta+2\frac{\theta^3}{3}+\ldots\ldots$$

But each term in the last development, except the first, is less than the sum of the corresponding terms in the first two developments.

Hence $\tan(\theta-x)+\tan(\theta+x)>2\tan\theta$. Hence $bb'>aa'$.

A geometrical proof is also easy.

In case half the angle at P is $>APD$, the triangle, as above, would not lie entirely within ABC. In that case make angle $APc=$the given angle 2θ, and APc is the triangle required.

The maximum triangle is formed by drawing from P a line PC to the farther vertex of the farthest side and making $CPE=2\theta$. If E falls on CB prolonged, then CPB is the required triangle. It will be observed that if $PB>PA$ triangle $CPB>CPA$.

MECHANICS.

75. Proposed by B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A particle, P, is held in a bent tube by two forces directed towards two fixed points, H and S. Show that the equation of the tube is $PS.PH=k^2$, if the forces are μ/PS and μ/PH.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Let $PS=r$, $PH=r'$, $\mu/PS=f$, $\mu/PH=f'$.

By the principle of virtual work we have for equilibrium,

$$fdr+f'dr'=0, \text{ but } f/f'=r'/r \text{ or } f'r'=fr.$$

Dividing $fdr=-f'dr'$ by $fr=f'r'$ we get

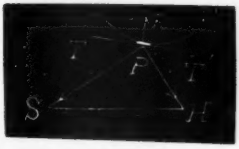
$$dr/r=-dr'/r' \text{ or } rdr'+r'dr=0.$$

Integrating, we get $rr'=$a constant$=k^2$.

$\therefore PS.PH=k^2$.

IV. Solution by GEORGE LILLEY, Ph.D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

Let P be any position of the particle, TT' the tangent to the tube at P, $\angle TPS=\phi$, $\angle TPH=\phi'$, $\angle PSH=\theta$, $\angle PHS=\theta'$, $PS=r$ and $PH=r'$.



Resolve along TT', $\frac{\mu}{r}\cos\phi+\frac{\mu}{r'}\cos\phi'=0$.

But, $\tan\phi=r\frac{d\theta}{dr}$; hence, $\cos\phi=\frac{dr}{ds}$, where ds is element length of the tube. Also $\cos\phi'=\frac{dr}{ds}$.

Therefore, $\frac{dr}{2r}+\frac{dr'}{2r'}=0$.

Integrating, $\log\sqrt{(rr')}+c=0$, where c depends on known values of r and r'.

Therefore, $SP.PH=k^2$.

Or thus: By the method of virtual work,

$Fdr+F'dr'=0$, where $F=\frac{\mu}{SP}$ and $F'=\frac{\mu}{HP}$.

Thus form the differential equation and solve it as above.

V. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

Let P be the particle, S and H the fixed points. Denote PS, PH, and SH by h, s, and $2m$, respectively. Take the origin of rectangular axes midway between S and H, the x-axis lying along the line SH. Let the resultant force

acting at $P(x, y)$ intersect Ox at A, and make angles α, β, θ, with PS, PH, and AH, respectively.

Since $\frac{\mu}{SP} : \frac{\mu}{HP} = \sin\beta : \sin\alpha$, $HP : SP = \sin\beta : \sin\alpha$,

$$\text{or } \frac{\sin\alpha}{\sin\beta} = \frac{h}{s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Also $\frac{s.\sin\beta}{h.\sin\alpha} = \frac{AH}{AS} = \frac{m-OA}{m+OA}$.

Substitute from (1) and solve for OA, obtaining

$$OA = \frac{h^2-s^2}{h^2+s^2}m. \quad \text{Tan}\theta = \frac{y}{x-OA} = \frac{y}{x-\frac{h^2-s^2}{h^2+s^2}m} = \frac{y^3+x^2y+m^2y}{x^3+xy^2-m^2x}.$$

Since PA is normal to the tube, the differential equation of the curve is

$$\frac{dy}{dx} = -\frac{x^3+xy^2-m^2x}{y^3+x^2y+m^2y}.$$

Integrating, $y^4+x^4+2x^2y^2+2m^2y^2-2m^2x^2=c$.
Adding m^4 to both members, and factoring,

$$[y^2+(m+x)^2][y^2+(m-x)^2]=c+m^4, \text{ or } SP^2.HP^2=c+m^4=k^4,$$

from which $SP.HP=k^2$.

VI. Solution by R. E. GAINES, Professor of Mathematics, Richmond College, Richmond, Va.

Denote PS and PH by r and r', respectively, and these may be taken as the "bipunctual coördinates" of P. Then it is easy to show that

$$\frac{dr'}{ds} = \cos\psi \text{ and } \frac{dr}{ds} = -\cos\varphi.$$

$$\therefore \frac{dr'}{dr} = -\frac{\cos\psi}{\cos\varphi}.$$

Now resolving forces along the tangent at P we have,

$$\frac{\mu}{r}\cos\varphi = \frac{\mu}{r'}\cos\psi \text{ or } \frac{1}{r} - \frac{1}{r'}\frac{\cos\psi}{\cos\varphi} = 0.$$

$$\therefore \frac{1}{r} + \frac{1}{r'}\frac{dr'}{dr} = 0.$$

$$\therefore \log r + \log r' = \log k^2, \quad \therefore rr' = k^2.$$

If the forces had been $\mu f(r)$ and $\mu F(r')$ we could get the form of the curve by integrating

$$f(r)+F(r')\frac{dr'}{dr}=0.$$

76. Proposed by JAMES F. LAWRENCE, Classical Sophomore, Drury College, Springfield, Mo.

An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If x' be the horizontal space advanced by the incline plane, x the part of the plane rolled over by the sphere, prove that $(M+m)x'=mx\cos\alpha$, $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$, where α is the inclination of the plane. [From *Routh's Elementary Rigid Dynamics*, page 126.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let F=friction of the sphere and plane, R=their mutual reaction, θ=the angle through which the sphere has rotated from the beginning of motion, y=the vertical distance of the center of the sphere from the horizontal plane, x_1=the corresponding abscissa, h and k the initial values of x and y, respectively, and a= the radius of the sphere.

For the motion of the sphere, resolving horizontally and vertically, and taking moments about the center of the sphere,

$$m\frac{d^2x_1}{dt^2}=F\cos\alpha-R\sin\alpha.....(1),\quad m\frac{d^2y}{dt^2}=F\sin\alpha+R\cos\alpha-mg.....(2),$$

$$\tfrac{2}{5}ma^2\frac{d^2\theta}{dt^2}=aF.....(3).$$

For the horizontal motion of the plane,

$$M\frac{d^2x'}{dt^2}=-F\cos\alpha+R\sin\alpha.....(4).$$

Also, $x_1=h+x'-a\theta\cos\alpha.....(5)$, $y=k-a\theta\sin\alpha.....(6)$.

From (5) and (1), $m\frac{d^2x'}{dt^2}-ma\cos\alpha\frac{d^2\theta}{dt^2}=F\cos\alpha-R\sin\alpha.....(7)$;

and from (6) and (2), $-ma\sin\alpha\frac{d^2\theta}{dt^2}=F\sin\alpha+R\cos\alpha-mg.....(8)$.

Eliminating F and R from (3), (7) and (8),

$$m\cos\alpha\frac{d^2x'}{dt^2}=\tfrac{7}{5}ma\frac{d^2\theta}{dt^2}-mg\sin\alpha.....(9).$$

Integrating (9), noticing that when $t=0$, $\frac{dx^2}{dt}=0$, $\frac{d\theta}{dt}=0$, and $x'=0$, $\theta=0$,

$$m\cos\alpha.x'=\tfrac{7}{5}ma\theta-\tfrac{1}{2}mg\sin\alpha.t^2 \quad \ldots\ldots(10).$$

Again, eliminating F and R from (1) and (4),

$$m\frac{d^2x_1}{dt^2}+M\frac{d^2x'}{dt^2}=0\ldots(11). \quad \text{But from (5), } m\frac{d^2x_1}{dt^2}=-ma\cos\alpha\frac{d^2\theta}{dt^2}\ldots(12).$$

$$(11)-(12) \text{ gives } ma\theta=\frac{(M+m)x'}{\cos\alpha}\ldots\ldots(13).$$

$$(13) \text{ in } (10) \text{ gives } x'=\frac{5m\sin\alpha\cos\alpha}{7(M+m)-5m\cos^2\alpha}\frac{gt^2}{2}\ldots\ldots(14).$$

$$(14) \text{ in } (13) \text{ gives } a\theta=\frac{5(M+m)\sin\alpha}{7(M+m)-5m\cos^2\alpha}\ldots\ldots(15).$$

Since $x=a\theta$, $\tfrac{7}{5}x-x'\cos\alpha=\tfrac{1}{2}gt^2\sin\alpha$..(16), and (13) is $(M+m)x'=mx\cos\alpha$..(17).

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, Univerility of Mississippi, P. O., University, Miss.

The horizontal component of the mutual action between the sphere and the inclined plane imparts to M the acceleration $2x'/t^2$ and to m the acceleration $\frac{2(x\cos\alpha-x')}{t^2}$. The forces producing these opposite motions being equal,

$$M\frac{2x'}{t^2}=m\frac{2(x\cos\alpha-x')}{t^2}.$$

From this $(M+m)x'=mx\cos\alpha$ . . . . . (1).

The principal of *vis viva* gives

$$2mgx\sin\alpha=M\frac{4x'^2}{t^2}+m\frac{4(x\cos\alpha-x')^2}{t^2}+m\frac{4x^2\sin^2\alpha}{t^2}+mk^2\left(\frac{d\theta}{dt}\right)^2$$

in which the first member is twice the work done by gravity, the first term of the second member has reference to the plane, the second term to the horizontal motion of the sphere, the third to its vertical motion, and the last to its rotation, $d\theta/dt$ being its angular velocity, and k^2 its radius of gyration about a diameter.

The motion of the sphere down the plane being one of pure rolling, $a(d\theta/dt)=2x/t$. (a=radius of sphere).

Substitute $2x/at$ for $d\theta/dt$, and $\frac{2}{5}a^2$ for k^2, and reduce, obtaining

$$\tfrac{1}{2}mgt^2x\sin\alpha=(M+m)x'^2+\tfrac{7}{5}mx^2-2mxx'\cos\alpha.$$

Using (1), this reduces to $\frac{1}{2}gt^2\sin\alpha=\frac{7}{5}x-x'\cos\alpha$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and CHARLES E. MYERS, Canton, O.

Take A as origin. Let (k, l), (y, z) be the coördinates of the center of the sphere when $t=0$ and when $t=t$, respectively, R, the reaction of the plane, P, the friction, a, the radius of the sphere, $\angle DOE=\theta$. Then $AA'=x'$ and $a\theta=x$.

For the sphere the equations of motion are

$\frac{2}{5}ma^2(d^2\theta/dt^2)=aP$.....(1). $m(d^2y/dt^2)=P\cos\alpha-R\sin\alpha$.....(2).

$m(d^2z/dt^2)=P\sin\alpha+R\cos\alpha-mg$.....(3).

For the plane, $M(d^2x'/dt^2)=-P\cos\alpha+R\sin\alpha$.....(4).

By geometry, $y=k+x'-a\theta\cos\alpha$, $z=l-a\theta\sin\alpha$.....(5, 6).

From (5) and (6) we get $d^2y/dt^2=d^2x'/dt^2-a\cos\alpha(d^2\theta/dt^2)$.....(7).

$d^2z/dt^2=-a\sin\alpha(d^2\theta/dt^2)$.....(8).

(7)$\cos\alpha$+(8)$\sin\alpha$ gives

$\cos\alpha(d^2y/dt^2)+\sin\alpha(d^2z/dt^2)=\cos\alpha(d^2x'/dt^2)-a(d^2\theta/dt^2)$ ....(9).

(2)$\cos\alpha$+(3)$\sin\alpha$ gives $m\cos\alpha(d^2y/dt^2)+m\sin\alpha(d^2x/dt^2)=P-mg\sin\alpha$.....(10).

From (9) and (10), $m\cos\alpha(d^2x'/dt^2)-ma(d^2\theta/dt^2)=P-mg\sin\alpha$.....(11).

(2)+(4) gives $m(d^2y/dt^2)+M(d^2x'/dt^2)=0$..... .....(12).

From (12) and (7) we get $(M+m)(d^2x'/dt^2=ma\cos\alpha(d^2\theta/dt^2)$... ..(13).

The value of P from (1) in (11) gives

$\frac{7}{5}a(d^2\theta/dt^2)-\cos\alpha(d^2x'/dt^2)=g\sin\alpha$.....(14).

Integrating (13) and (14) and remembering that when $t=0$, $x=0$ and $\theta=0$ we get

$(M+m)x'=ma\theta\cos\alpha$ or $(M+m)x'=mx\cos\alpha$,

$\frac{7}{5}a\theta-x'\cos\alpha=\frac{1}{2}gt\sin\alpha$ or $\frac{7}{5}x-x'\cos\alpha=\frac{1}{2}gt^2\sin\alpha$.

DIOPHANTINE ANALYSIS.

71. Proposed by A. H. BELL, Hillsboro, Ill.

Find five numbers such that the product of any two plus 1 will equal a square.

I. Solution by CHARLES C. CROSS, Libertytown, Md.

On page 301, Vol. V, of MONTHLY I found four general numbers to be: m, $n^2-1+(m-1)(n-1)^2$, $n(mn+2)$, and $4m(mn^2-mn+2n-1)^2+4(mn^2-mn+2n-1)$.

If we take m and m for the first two numbers, then $m.m+1=\square=y^2$(say) $=(y-s)^2=y^2-2sy+s^2$.

$\therefore y=(s^2+1)/2s$; whence $m=(s^2-1)/2s$.

Let $s=2$, then $m=\frac{3}{4}$. Take $n=2$ and the numbers are $\frac{3}{4}$, $\frac{3}{4}$, $\frac{11}{4}$, 7, and $\frac{345}{4}$.

Take $n=3$ and the numbers are $\frac{3}{4}$, $\frac{3}{4}$, 7, $\frac{51}{4}$, and $\frac{123}{4}$.

Let $s=3$, then $m=\frac{4}{3}$. Take $n=3$ and the numbers are $\frac{4}{3}$, $\frac{4}{3}$, $\frac{28}{3}$, $\frac{260}{3}$, etc.

II. Solution by Euler, found in the MSS. of Lagrange by Legendre.

Let the numbers be x, y, z, u, v. To find the first three, such that $xy+1$, $xz+1$, and $yz+1$ shall be squares.

Take any square, l^2, and decompose l^2-1 into any two unequal factors, x and y; then $xy=l^2-1$, and take $z=2l+x+y$.

$\therefore xy+1=l^2$, $xz+1=x(2l+x+y)+1=(l+x)^2$

and $yz+1=y(2l+x+y)+1=(l+y)^2$.

Now take $u=4l(l+x)(l+y)$.....(1).

$\therefore zu+1=4lx(l+x)(l+y)+(l^2-xy)^2=(l^2+2lx+xy)^2$,

$yu+1=4ly(l+x)(l+y)+(l^2-xy)=(l^2+2ly+xy)^2$,

$zu+1=4l(l+x)(l+y)(2l+x+y)+(l^2-xy)^2=[3l^2+2l(x+y)+xy]^2$.

To find the fifth number, v, we make an equation of the fourth degree from the roots x, y, z, and u, thus $a^4-pa^3+qa^2-ra+s=0$.....(2), from which $v=\frac{4r+2p(s+1)}{(s-1)^2}$.....(3).

To verify this, $va+1=\square$, $\therefore 4ra+2p(s+1)a+(s-1)^2$ is a square, but in (2) $4ra=4a^4-4pa^3+4qa+4s$.

Now $4a^4-4pa^3+4qa+2p(s+1)a+(s+1)^2=\square$ .....(4).

When the condition $\frac{1}{4}p^2=q+s+1$.....(5) is satisfied the demonstration is completed.

From (2) $p=x+y+z+u$,

$q=xy+(x+y)(z+u)+zu$,

$p^2-4q=(x+y-z-u)^2-4xy-4zu=(2l+u)^2-4(l^2-1)-4zu$,

$4(s+1)=4xyzu+4=4zu(l^2-1)+4$.

$\therefore p^2-4q-4(s+1)=u(u+4l-4l^2z)$.

But in (1) we have $u+4l-4l^2z=0$.

$\therefore p^2-4q-4(s+1)=0$ .....(6) and the condition is proved. This value of $4qa$ in (4) gives the general value

$$va+1=\frac{4a^4-4pa^3+p^2a^2-4a^2(s+1)+2p(s+1)a+(s+1)^2}{(s-1)^2}=\square$$

$$=\frac{(2a^2-pa-s-1)^2}{(s-1)^2}.$$

$$\therefore vx+1=\frac{(2x^2-px-s-1)^2}{(s-1)^2};\ vy+1=\frac{(2y^2-py-s-1)^2}{(s-1)^2};$$

$$vz+1=\frac{(2z^2-pz-s-1)^2}{(s-1)^2};\ vu+1=\frac{(2u^2-pu-s-1)^2}{(s-1)^2}.$$

And the five numbers are $xy=l^2-1$, x, y, $2l+x+y$, $4l(l+x)(l+y)$, and $v=\frac{2r+2p(s+1)}{(s-1)^2}$.

$r=xyz+xyu+xzu+yzu$; $s=xyzu$ and $p=x+y+z+u$.

If $l^2=4$ then the five numbers$=1$, 3, 8, 120, and $\frac{777480}{(2879)^2}$.

If $l^2=25$ then the five numbers$=1$, 24, 35, 3480, not carried out.
2, 12, 24, 2380, not carried out.
3, 8, 21, 2080, not carried out.
4, 6, 20, 1980, $\frac{3822388020}{(950399)^2}$.

The Hillsboro Mathematical Club have solved for an integral value by extending the series, without result. A. H. BELL.

AVERAGE AND PROBABILITY.

65. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What is the average rate of the sun's motion in declination from the equator to the solstices ?

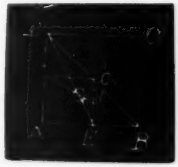
No solution of this problem has been received.

66. Proposed by REV. W. ALLEN WHITWORTH, A. M.

A rod 9 feet long is to be divided into three parts, of which A is to have the largest, B the next, and C the smallest. If the two fractures are made at random, A's, B's, and C's expectations will be, respectively, 66, 30, and 12 inches. But, if one fracture be made at random and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Let $AB=AD=108=a$, $AE=AF=\frac{1}{3}a$.



I. To find the mean value of the least part, we find the values or limits of x, y by restricting the point to the area GOH. The limits for the denominator are given by restricting the point to GOB.

$$\therefore L=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{3}(a-x)}^{x} x\,dx\,dy}{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{3}(a-x)}^{x} dx\,dy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{3}(a-x)}^{a-x} dx\,dy}=\frac{\frac{a^3}{108}}{\frac{a^2}{12}}=\frac{1}{9}a=12 \text{ inches}.$$

For the mean value of the greatest part the limits are the same for both integrations :

$$\therefore G=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{3}(a-x)}^{x} x\,dx\,dy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{3}(a-x)}^{a-x} x\,dx\,dy}{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{3}(a-x)}^{x} dx\,dy+\int_{\frac{1}{2}a}^{a}\int_{\frac{1}{3}(a-x)}^{a-x} dx\,dy}=\frac{\frac{11a^3}{216}}{\frac{a^2}{12}}=\tfrac{11}{18}a=66 \text{ inches.}$$

$\therefore M=a-(\frac{1}{6}a+\frac{1}{18}a)=\frac{5}{18}a=30$ inches.

II. For the mean value of the least part, the limits for x are found by restricting the point to the area ADG, for y, by restricting the point to $AEHG$ and doubling.

$$\therefore L=\frac{\int_{0}^{\frac{1}{3}a}\int_{x}^{a-2x} x\,dx\,dy+2\left[\int_{0}^{\frac{1}{3}a}\int_{y}^{\frac{1}{2}(a-y)} y\,dy\,dx+\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2x}^{\frac{1}{2}(a-x)} y\,dx\,dy\right]}{\int_{0}^{\frac{1}{3}a}\int_{0}^{a-x} dx\,dy},$$

$$=\left[\frac{a^3}{54}+\frac{a^3}{54}+\frac{7a^3}{864}\right]\div\frac{3a^2}{8}=\frac{13a}{108}=13 \text{ inches.}$$

For the mean value of the greatest part, x is given by restricting the point to GOE, y, by $DFGO$ and doubling.

$$\therefore G=\frac{\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2x}^{x} x\,dx\,dy+2\left[\int_{\frac{1}{2}a}^{a}\int_{0}^{a-y} y\,dy\,dx+\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{a-2y}^{y} y\,dy\,dx\right]}{\int_{0}^{\frac{1}{3}a}\int_{0}^{a-x} dx\,dy},$$

$$=\left[\frac{a^3}{54}+\frac{a^3}{6}+\frac{a^3}{27}\right]\div\frac{3a^2}{8}=\frac{16a}{27}=64 \text{ inches.}$$

$$\therefore M=a-\left[\frac{13a}{108}+\frac{16a}{27}\right]=\frac{31a}{108}=31 \text{ inches.}$$

67. Proposed by B. F. FINKEL, A.M.,M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A person writes n letters and addresses n envelopes; if the letters are placed in the envelopes at random, what is the probability that every letter goes wrong? [From *Hall and Knight's Higher Algebra.*]

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let a, b, c, ...n represent the letters.

The number of all possible cases is $1.2.3..... \; n=n\,!=u_n$, say.

The number of cases, the first letter a, and envelope used is u_{n-1}.

The number of cases, the second letter and envelope used (without the first a) is $u_{n-1}-u_{n-2}$.

The number of cases, the third letter and envelope used (without the first having been a or the second b) is $u_{n-1}-2u_{n-2}+u_{n-3}$.

Thus we have successively,

$$u_{n-1},$$
$$u_{n-1}-u_{n-2},$$
$$u_{n-1}-2u_{n-2}+u_{n-3},$$
$$u_{n-1}-3u_{n-2}+3u_{n-3}-u_{n-4},$$
$$\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots$$

The sum of these n lines is

$$S=u_n\left[\frac{1}{1}-\frac{1}{2\,!}+\frac{1}{3\,!}-\frac{1}{4\,!}+\ldots\ldots-\frac{(-1)^n}{n\,!}\right].$$

C, the required chance, is $1-S/u_n$.

$$\therefore\ C=\frac{1}{2\,!}-\frac{1}{3\,!}+\frac{1}{4\,!}-\ldots\ldots+\frac{(-1)^n}{n\,!}.$$

[NOTE. A solution of problem 61 will be published in the next issue. If any one will send us a solution of problem 65, it will also appear in the next number. ED. F.]

67a. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

A witness in court who undertook to recognize the signature of an individual failed four times in succession. What is the probability that he was correct the fifth time? An actual occurance.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let p represent the probability of his failing again the fifth time, then we have,

$$p=\int_0^1 x^5\,dx \div \int_0^1 x^4\,dx=\tfrac{5}{6}.$$

$\therefore$ The probability of his being correct the fifth time is $\frac{1}{6}$.

II. Solution by the PROPOSER.

In the absence of other evidence we are compelled to judge of the credibility of the witness by his present testimony.

If x represents his credibility, the *apriori* probability that he would have testified falsely four times in succession is $(1-x)^4$.

Hence the chance that x had any particular value is $(1-x)^4dx/\int_0^1(1-x)^4dx$ $=5(1-x)^4dx$, and the chance of the event if x has this particular value is x.

Hence the chance of the event through any particular value of x is $5x(1-x)^4dx$ and the chance through all possible values is $\int_0^1 5x(1-x)^4dx=\frac{1}{6}$.

That is, the chances are 5 to 1 against the correctness of his testimony.

MISCELLANEOUS.

68. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Find the locus of the vertex of the cone enveloping the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ so that the plane of contact will constantly touch $x^2+y^2+z^2=r^2$.

I. Solution by ELMER SCHUYLER, High Bridge, N. J.

From Aldis's *Solid Geometry*, equation of plane of contact to ellipsoid is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1=0,\ \frac{xx'}{r^2}+\frac{yy'}{r^2}+\frac{zz'}{r^2}-1=0,$$

equation of tangent to circle, $x'^2+y'^2+z'^2=r^2$.

$\therefore$ Since the plane of contact is tangent to sphere,

$$\frac{x'}{r^2}=\frac{\alpha}{a^2},\quad \frac{y'}{r^2}=\frac{\beta}{b^2},\quad \frac{z'}{r^2}=\frac{\gamma}{c^2}\text{ and }\left(\frac{\alpha r^2}{a^2}\right)^2+\left(\frac{\beta r^2}{b^2}\right)^2+\left(\frac{\gamma r^2}{c^2}\right)^2=r^2,$$

or equation is

$$\left(\frac{\alpha r}{a^2}\right)^2+\left(\frac{\beta r}{b^2}\right)^2+\left(\frac{\gamma r}{c^2}\right)^2=1,$$

which locus is an ellipsoid similar to the given one but with ratio of axes as $\sqrt{r}:1$.

II. Solution by W. B. CARVER, Senior Class, Dickinson College, Carlisle, Pa.

Let (x', y', z') be a point. Then $b^2c^2x'x+a^2c^2y'y+a^2b^2z'z=a^2b^2c^2$ is the equation of the plane of contact of the cone whose vertex is at (x', y', z') with the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The condition that this plane touch the sphere $x^2+y^2+z^2=r^2$ is

$$b^4c^4x'^2+a^4c^4y'^2+a^4b^4z'^2=\frac{a^4b^4c^4}{r^2}.$$

Letting (x', y', z') move and x', y', and z' become variables, we have for the required locus

$$b^4c^4x^2+a^4c^4y^2+a^4b^4z^2=\frac{a^4b^4c^4}{r^2}\quad\text{or}\quad\frac{x^2}{a^4/r^2}+\frac{y^2}{b^4/r^2}+\frac{z^2}{c^4/r^2}=1,$$

which is the equation of an ellipsoid.

III. Solution by the PROPOSER.

If (x', y', z') be the vertex of the cone, the plane of contact is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}-1=0\ldots\ldots(1);$$

and the sphere being $x^2+y^2+z^2-r^2=0$.....(2), the condition that (2) is touched by (1) is

$$\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}-\frac{1}{r^2}=0.....(3).$$

a concentric ellipsoid.

69. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, P. O., Sebastopol, Cal.

Find the locus of a point equidistant from the circumferences of two fixed circles.

Solution by ELMER SCHUYLER, High Bridge, N. J.

Let radii be a and b, and $OO'=c$.

$OP=\sqrt{(x^2+y^2)}$.....(1), $O'P=\sqrt{[y^2+(c-x)^2]}$..... .....(2).

By condition, $O'P-OP=b-a$.

$\therefore \sqrt{[(c-x)^2+y^2]}-\sqrt{[x^2+y^2]}=b-a$.....(3).

Clearing of fractions gives us $[c^2-(b-a)^2-2cx]^2=4(b-a)^2(x^2+y^2)$..(4).

This is a conic section and an ellipse, hyperbola, parabola (or particular case) according as $(b-a)^2[(b-a)^2-c^2]>$, $<$, or $=0$.

NOTE ON RIGHT TRIANGLES.

Every right-angled triangle has two concealed roots. By three different combinations of the two roots, the three sides are formed. The longest side is the sum of the squares of the two roots. The second side is the difference of the squares of the two roots. The third side is twice the product of the two roots.

The perimeter is equal to twice the greater root multiplied by the sum of the two roots. The area is equal to the product of the two roots multiplied by the product of the sum and difference of the two roots.

A prime right-angled triangle is one whose sides are integral and cannot all be divided by the same number without a remainder. A prime triangle is the result of having one of the roots odd and one even. Exception.—If the even root is just twice the odd root, the resulting triangle will not be prime, as its sides will all be divisible by the square of the odd root.

If both the roots be odd or both even the sides of the triangle will be divisible by two and the triangle will not be prime. Any odd number may be one side of a prime triangle; in many cases the same odd number will serve as one side of several different prime triangles, as for example, 13, 12, 5, and 13, 84, 85. Any even number divisible by 4 can be one side of a prime triangle as a prime triangle has always one even side. The digit, 2, must be a factor twice, or both the roots will be odd numbers. The same even number may be a side of several prime triangles, as 12, 13, 5, and 12, 35, 37.

A prime right-angled triangle has two of its sides expressed by odd numbers. Find the sum and the difference of these. Each will be the double of a perfect square. The square root of one-half the sum will be the greater root of

the triangle, and the square root of one-half the difference will be the lesser root of the triangle.

Next take the hypotenuse and the even side of any prime right-angled triangle. The sum and the difference will each be a perfect square number, and their square roots will be the sum and the difference of the two roots of the triangle.

A given area, or a given perimeter, can belong to but one prime right-angled triangle, and either the area or the perimeter being given it is easy to find the other dimensions. In case of one side only being given it has been shown that the given side may belong to several different triangles, but all of them are easily found.

C. W. SHEDD.

Columbus, Miss.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

110. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.**

By measuring with a yard $m=12\frac{1}{2}\%$ *too short*, my profits are $n=25\%$ of my sales. If my yard be $p=10\%$ *too long*, what per cent. of my sales will be my profits ?

111. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.**

By what per cent. of its original dimensions must a linear yard of steel rail, weighing 60 pounds, be increased so that it may weigh 75 pounds ?

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

ALGEBRA.

98. **Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

A and B agreed to reap a field of grain for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days; but B, who did not work as quickly as he expected, was obliged to call to his assistance C, an inferior workman, who worked the last two days, in consequence of which B received 3*s*. 9*d*. less than would otherwise have been due him. In what time could B and C each reap the field ? From *Milne's High School Algebra*.

99. **Proposed by C. H. JUDSON, Greenville, S. C.**

Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

*** Solutions of this problem should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

118. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

A picture b feet long hangs on a wall at an inclination θ to the wall, with its base a feet from the floor. How far from the wall should an admirer sit to see it to the best advantage, supposing the light to be equally distributed throughout the room ?

119. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

A sphere touches each of two straight lines which are inclined to each other at a right angle but do not meet; show that the locus of its center is an hyperbolic paraboloid.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

88. Proposed by JOHN M. ARNOLD, Crompton, R. I.

When a watch is wound up, the mainspring is closely coiled around a cylindrical piece called the hub of the barrel-arbor. When entirely run down the spring forms an annulus against the inner circumference of the barrel. Show that if the width of the annulus is a little more than one-fourth of the radius of the barrel, the spring will run the watch the greatest number of hours at one winding, the diameter of the hub being one-third the inside diameter of the barrel.

89. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Integrate the equation,

$$\frac{dy}{dx}+y\cos x=\frac{\sin 2x}{2}.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

87. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

> "He on his impious foes onward drove,
> Drove them before him to the bounds
> And crystal walls of Heaven; which opening wide
> Rolled inward and a spacious gap disclosed
> Into the wasteful deep; headlong themselves they threw
> Down from the verge of Heaven.
> Nine days they fell; Hell at last
> Yawning received them whole and on them closed."
>
> *Paradise Lost, Book VI.*

Assuming Hell to be the center of the earth and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

88. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Show that the equation to the trajectory is

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha},$$

and that v and a can be varied at pleasure, the projectile can in general be made to traverse any two given points in the same vertical plane with the point of projection. [Ex. 83, page 244, Deschanel's *Natural Philosophy*, Part I.]

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

75. Proposed by J. C. HAGLE, Professor of Civil Engineering, Agricultural and Mechanical College, College Station, Texas.

The water tank at the Nacogdoches River on the H. E. & W. T. Ry. is filled by a 3-inch pipe from a reservoir in which the water level is 6 feet above water in tank when full. The top diameter of tank is 17 feet, the bottom diameter is 19 feet, 8 inches, and the pipe projects 10 inches through the bottom. The depth is 13 feet, 6 inches. Find the time required to fill tank, taking the pipe as clean and free from sharp bends, except the right-angled one directly under tank. This bend is 12 feet below outlet of pipe, so that the total length of pipe is 1972 feet. Compare the result with the time of filling if the inlet pipe projected over top of tank instead of entering at the bottom.

76. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Show that

$$\log[x-a-b\sqrt{(-1)}]=\tfrac{1}{2}\log[(x-a)^2+b^2]-\sqrt{(-1)}\tan^{-1}\frac{b}{x-a},$$

Naperian logarithms being used.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

EDITORIALS.

Contributors are requested to send in select problems for the Departments of Arithmetic, Mechanics, and Average and Probability.

The *Annals of Mathematics*, published for the past fifteen years at the University of Virginia, is to be transferred to Harvard University with the close of the present volume.

Mathematics and the mathematical world have sustained a great loss in the death of Prof. Sophus Lie, which occured at Christiania, Monday, February 18. He was Professor of Geometry at the University of Leipzig from 1886 to 1898, and at the time of his death was Professor of Mathematics at the University of Christiania. We hope to be able to give a biographical sketch of Professor Lie in a future issue of the MONTHLY.

Cornell University announces a Course of Instruction during the Summer, session to be held July 5 to August 16. The following courses in Advanced Mathematics are offered : Advanced Integral Calculus, Prof. Wait ; Differential Equations, Dr. Murray ; Projective Geometry, Prof. Wait ; Theory of Functions of a Complex Variable, Dr. G. A. Miller ; Theory of Groups of a Finite Order, Dr. G. A. Miller ; Theory of Numbers, Dr. G. A. Miller.

BOOKS AND PERIODICALS.

Tables of Logarithms to Five Places of Decimals with Auxiliary Tables. Edited by Edwin S. Crawley, Ph. D., Assistant Professor of Mathematics in the University of Pennsylvania. 8vo. Cloth, xxxii+76 pages. Price, 75 cents. Published by the author.

These tables have been prepared to accompany Dr. Crawley's *Elements of Trigonometry* and we are told by the author that great care has been taken to secure accuracy.—the proof having been compared twice, number by number, with different standard tables, *e. g.* Vega's seven-place tables, seventy-fourth edition edited by W. L. Fischer, and Gauss's five-place tables, twentieth edition. In addition to this careful comparison, the method of differences was also applied as a further check. A table not usually found in a collection of tables of this sort, and one which is quite useful in many lines of mathematical work, is table VI in which every degree, minute and second from 0 to 180° is expressed in Radians. B. F. F.

A Short History of Astronomy. By Arthur Berry, M. A., Fellow and Assistant Tutor of King's College, Cambridge ; Fellow of University College, London. 8vo. Cloth, xxxii+440 pages. Price, $1.50. New York : Charles Scribner's Sons.

In this work we have an interesting account of the progress and development of Astronomy from the earliest time down to the present day, and presented in a form intelligible to readers having no special knowledge of either mathematics or Astronomy. Only the essence is here preserved. Little mention is made of Astronomical Instruments, it being held by the author that little pleasure or profit is derived from a written description of scientific Instruments. A most valuable aud interesting feature of the work is the short biographical sketches of leading Astronomers, accompanied by an excellent portrait printed on heavy paper. Of the biographical sketches accompanied by portraits we mention Copernicus, Tycho Brahe, Galilei, Kepler, Newton, Bradley, Lagrange and William Herschel. The numerous illustrations throughout the work are very good and the reproductions from original photographs are first class. The book is one of great interest to the Astronomer as well as to the ordinary student and amateur. B. F. F.

Mathematical Essays and Recreations. By Herman Schubert, Professor of Mathematics in the Johanneum, Hamburg, Germany. Translated from the German by Thomas J. McCormack. Red Cloth, 148 pages. Price, 75 cents. Chicago : The Open Court Publishing Co.

The following subjects are discussed in a popular though scientific manner: (1) "The Definition and Notion of Number," (2) "Monism in Arithmetic," (3) "On the Nature of Mathematical Knowledge," (4) "Magic Squares," (5) "The Fourth Dimension," (6) "The History of the Squaring of the Circle."

The first three articles of the book are concerned with the construction of arithmetic as a monistic science. Number is defined as the result of counting. Fractional numbers, complex numbers, negative numbers, irrational numbers, imaginary numbers are all extensions of primitive results, made according to what Hankel calls "the principle of permanence," and Schubert the "principle of no exception." Arithmetic thus takes the general shape of a system of logical forms having consistency and coherency among themselves. Professor Schubert being one of the most successful teachers in Germany, his sketch on monistic arithmetic will be found exceedingly interesting and suggestive. Any other theory of the number concept seems to me absolutely untenable.

The article on "Fourth Dimension" is a clear and easily understood exposition of the apparently mysterious subject. Professor Schubert clearly shows what is meant by dimensions as used in science and what is the legitimate function of a fourth dimension in mathematics. This article alone is worth the price of the book. The "History of the Squaring of the Circle" is full of interest from first to last, and is quite complete.

B. F. F.

The Elements of Physics. A Text-book for High Schools and Academies. By Alfred Payson Gage, Ph. D., Author of Principles of Physics, Introduction to Physics, etc. Revised Edition. 12mo. Half Leather, 381 pages. Introduction price, $1.12. New York and Chicago : Ginn & Co.

In bringing out the revision of this book, all the excellencies of the original work have been retained while many improvements in method of presentation have been introduced. Recent advances in the industrial applications of physical principles have received due attention. A large number of problems and practical exercises are furnished throughout the book. B. F. F.

A Text-book of General Astronomy for Colleges and Scientific Schools. By Charles A. Young, Ph. D., LL. L., Professor of Astronomy in Princeton University. Revised Edition. 8vo. Half Leather, 630 pages. Price, $2.50. Boston, U. S. A., and London : Ginn & Co.

In bringing out the revised edition of this book, the best book on astronomy has been improved. Our acquaintance with this work was made by its use for several years in the class-room, and we have found it very satisfactory in every particular. The revised edition embodies the new and important results which have been obtained during the last ten years, and thereby the continued popularity of the work is insured for sometime in the future. B. F. F.

A Text-book of Physics. By G. A. Wentworth, Author of a Series of Text-books in Mathematics, formerly Professor of Mathematics in Phillips Exeter Academy, and G. A. Hill, Author of Geometry for Beginners, formerly Assistant Professor of Physics in Harvard University, 12mo. Half Leather, 440 pages. Price for Introduction, $1.15. Boston and Chicago : Ginn & Co.

This work aims to give a rational explanation of the more important physical phenomena, and to prepare the way for further investigation and study of physical sciences. The book has many points of excellence to commend it to public favor. B. F. F.

ERRATA.

Page 42, line 9, for "$\angle CEM$" read $\angle CFM$.
line 12, for "$A-60°$" read $B-60°$.
line 22, for "angle FCL" read angle FCM.
line 23, for "angle EMC" read angle FMC.
line 24, for "$x+A+C-60°$" read $x+B+C-60°$.
line 24, for "$A+x+y$" read $B+x+y$.

Vol. V., No. 10, page 231, line 7, of solution, for "$\triangle ABC=\triangle CPB$" read $\triangle ABC+\triangle CPB$.

Same, line 8, for "$\frac{r_1 b}{2}=\frac{r_1 c}{2}$" read $\frac{r_1 b}{2}+\frac{r_1 c}{2}$.

Same, line 11, for "$=\frac{r_2 a}{2}$" read $+\frac{r_2 a}{2}$.

SOPHUS LIE.

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BIOGRAPHY.

SOPHUS LIE.

BY GEORGE BRUCE HALSTED.

ON the eighteenth of February, 1899, the greatest mathematician in the world, Sophus Lie, died at Christiania in Norway.

He was essentially a geometer, though applying his splendid powers of space creation to questions of analysis. From Lie comes the idea that every system of geometry is characterized by its group.

In ordinary geometry a surface is a locus of points ; in Lie's *Kugel-geometrie* it is the aggregate of spheres touching this surface. By a simple correlation of this sphere-geometry with Pluecker's line-geometry, Lie reached results as unexpected as elegant. The transition from this line-geometry to this sphere-geometry was an example of contact-transformations.

Now contact-transformations find application in the theory of partial differential equations, whereby this theory is vastly clarified. Old problems were settled as sweepingly as new problems were created and solved. Again, with his *Theorie der Transformationsgruppen*, Lie changed the very face and fashion of modern mathematics.

A magnificent application of his theory of continuous groups is to the general problem of non-Euclidean geometry as formulated by Helmholtz. To this was awarded the great Lobachevski Prize. Not even this award could sufficient-

ly emphasize the epoch-making importance of Lie's work in the evolution of geometry.

Moreover, the foundations of all philosophy are involved. To know the non-Euclidean geometry involves abandonment of the position that axioms as to their concrete content are necessities of the inner intuition ; likewise abandonment of the position that axioms are derivable from experience alone.

Lie said that in the whole of modern mathematics the weightiest part is the theory of differential equations, and, true to this conviction, it has always been his aim to deepen and advance this theory.

Now it may justly be maintained that in his theory of transformation groups Lie has himself created the most important of the newer departments of mathematics.

By the introduction of his concept of continuous groups of transformations he put the isolated integration theories of former mathematicians upon a common basis.

The masterly reach of Lie's genius is illustrated by his encompassment of the fundamentally important theory of differential invariants associated with the English names Cayley, Cockle, Sylvester, Forsyth.

Thirteen years ago Sylvester announced his conception of 'Reciprocants,' a body of differential invariants not for a group, but for a mere interchange of variables. A number of Englishmen thereupon took up investigations about orthogonal, linear and projective groups, groups in whose transformations interchanges of variables occur as particular cases, and whose differential invariants are consequently classes of reciprocants, and of the analogues of reciprocants, when more variables than two are considered.

Now all these investigations were long subsequent to Lie's consideration of the groups in question as leading cases of a general conception. Thus they were merely secondary investigations !

Again the theory of complex numbers appears as a part of the great 'Theorie der Transformationsgruppen.' Indeed, this continent of 'transformations' opened up and penetrated with such giant steps by Lie represents the most remarkable advance which mathematics in all its entirety has made in this latter part of the century.

Sophus Lie it was who made prominent the importance of the notion of group, and gave the present form to the theory of continuous groups. This idea, like a brilliant dye, has now so permeated the whole fabric of mathematics that Poincaré actually finds that in Euclid 'the idea of the group was potentially pre-existent,' and that he had 'some obscure instinct for it, without reaching a distinct notion of it.' Thus the last shall be first and the first last.

In personal character Lie was our ideal of a genius, approachable, outspoken, unconventional, yet at times fierce, intractable.

His work is cut short ; his influence, his fame, will broaden, will tower from day to day.